Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 5

P. Reany

December 14, 2022

Abstract

This paper contains my notes on Lecture Five of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes.

1 A bit of review

The electron is a spin-$\frac{1}{2}$ particle and it obeys the Pauli Exclusion Principle. We can put two electrons into the ground state of the Helium atom, one spin up, the other spin down, and that satsifies the Pauli Exclusion Principle. (By the way, these two electrons are entangled.) In the Lithium atom, two electrons fill the ground state and a third electron must be placed in a higher state.

> Half-integer spin particles always obey the Pauli Exclusion Principle. The integer-spin particles never obey it.

We can declare an orthonormal family of states by this notation: $|\, x_1\, x_2\, x_3 \,\rangle$. Now, the wave function of the state described is the inner product of the vector $|\, x_1\, x_2\, x_3 \,\rangle$ with the state vector $|\, \psi \,\rangle$:

$$\langle\, x_1 \dots x_n \mid \psi \,\rangle = \psi(x_1 \dots x_n)\,. \tag{1}$$

So, to get the probability that particle 1 is at x_1, and particle 2 is at x_2, continued all the way up to particle n at x_n, we complex square this last expression: $\psi^*(x_1, \dots, x_n)\psi(x_1, \dots, x_n)$.

Ignoring spin, we investigate otherwise identical particles, such as electrons. To date, experiment suggests that we declare that a wave function under any permutation is invariant. In other words, let $\mathcal{P}$ be any permutation of indices in (1), then

$$\mathcal{P}\,|\, x_1, \dots, x_n \,\rangle = |\, x_{i_1}, \dots, x_{i_n} \,\rangle = e^{i\phi}\,|\, x_1, \dots, x_n \,\rangle\ , \tag{2}$$

where ϕ is a fixed real value. The complex number $e^{i\phi}$ is called a *phase* and it will not affect the probabilities because it will be cancelled out by the additional factor of $e^{i\phi*} = e^{-i\phi}$ when calculating the probability..

Let's look at the case of just two particles.

$$|\, x_1, x_2 \,\rangle = e^{i\phi}\,|\, x_2, x_1 \,\rangle\ . \tag{3}$$

But if we interchange the particles once more, we must arrive at the original state. From this insight, we conclude that

$$|\, x_1, x_2 \,\rangle = \pm\,|\, x_2, x_1 \,\rangle\ , \tag{4}$$

where the + is for bosons and the − is for fermions. On going from state vectors to wave functions, we have that

$$\langle\, x_1, x_2 \mid \psi \,\rangle = \psi(x_1, x_2)\,, \tag{5a}$$
$$\langle\, x_2, x_1 \mid \psi \,\rangle = \psi(x_2, x_1)\,. \tag{5b}$$

Hence, the property of the state vectors becomes the property of the corresponding wave functions.

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1)\,. \tag{6}$$

Imagine that we place two bosons in the same state ψ_0

$$\psi_0(x_1)\psi_0(x_2) = \psi_0(x_2)\psi_0(x_1)\,, \tag{7}$$

which shows that the interchange of the particles is symmetric.

Next, let's put bosons in different states ψ_0 and ψ_1:

$$\psi_0(x_1)\psi_1(x_2) \longrightarrow \psi_0(x_2)\psi_1(x_1)\,, \tag{8}$$

where we have no knowledge how to compare these two two-particle states. However, we can rectify this state on nonknowledge by creating a wavefunction that is symmetric under interchange of particles, namely,

$$\psi_0(x_1)\psi_1(x_2) + \psi_0(x_2)\psi_1(x_1)\,. \tag{9}$$

Clearly, this expression is invariant under interchange of bosons 1 and 2.

So, it should be obvious that to construct a wave function that is invariant for interchange of fermions, we only need to proffer a similar wavefunction, but this one much change sign under exchange of particles.

$$\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)\,. \tag{10}$$

So, what happens if the particles in state 2 are moved to state 1? In the case of the symmetric wave function we get

$$2\psi_0(x_1)\psi_0(x_2)\,, \tag{11}$$

which is fine for bosons, and in the case of the antisymmetric combination, we get

$$\psi_0(x_1)\psi_0(x_2) - \psi_0(x_2)\psi_0(x_1) = 0\,, \tag{12}$$

which for fermions means that this combined state is not permissible.

We should add the spin state of the particle to its state information, such as for particle 1,

$$\psi(x_1) \longrightarrow \psi(x_1, \sigma_z)\,, \tag{13}$$

where σ_z will take the value of either ± 1. For the wave function of a multiparticle system, we get

$$\psi(x_1\sigma_{z1}, x_2\sigma_{z2}, \ldots)\,. \tag{14}$$

So, now what happens when we exchange, say, the first two components? Well, for bosons, the wave function stays the same, but for fermions, it changes sign.

How does the total angular momentum J affect the wave function?

J is the generator of rotations, so when it acts on a state, it differentiates it with respect to an angle. Specifically,

$$J_z \,|\, \psi \,\rangle = -i\frac{\partial}{\partial\theta}\,|\, \psi \,\rangle\;, \tag{15}$$

which is a rotation about the z-axis. The eigenvalue version of this is

$$-i\frac{\partial}{\partial\theta}\,|\,\psi\,\rangle = m\,|\,\psi\,\rangle\ , \tag{16}$$

which has solution

$$|\,\psi(\theta)\,\rangle = e^{im\theta}\,|\,\psi_0\,\rangle\ . \tag{17}$$

When m is an integer and we rotate the wavefunction by 2π, we return to the start. However, if m is a half-integer, when we rotate by 2π, the wave function changes sign.

Note: Experiments involving coffee and dress belts are left to the reader.

In Fig. 1, we find the setup for an experiment on an electron in a prepared state of spin up, held in place by a magnetic field in the cavity. Then the box will be rotated in a full circle about an axis through the box. (The reason to rotate the box slowly is to ensure that the magnetic field continuously aligns the electron spin with the magnetic field.) We will then compare the interference results of this experiment with a similar experiment in which the electron box is not rotated prior to the release of the electron. The results of both experiments should be statistically the same.

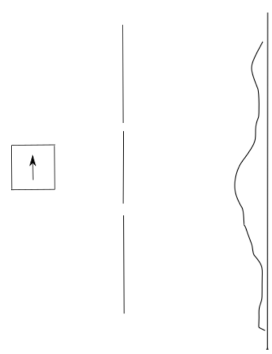

Figure 1. The electron is placed within a cavity in prepared state as spin up. Then the cavity/box is slowly rotated by 2π about an axis through the box.

Fig. 2 shows an experiment in which the incoming electron is forced through a beam splitter, which has the effect of forcing part of the electron's wave function into the upper box, and the rest into the lower box – in other words, a superposition. Both boxes have magnetic fields points up in them. Now, before we release the electron by opening up both boxes, we will rotate one of them by 2π, as we had done before. The combined wave function before the rotation is $\psi_1 + \psi_2$, whereas, the waved function after the rotation is $\psi_1 - \psi_2$. In this case, the interference pattern will be very different between the two of them. The upshot of this is that we can by experimental means detect the rotation of an electron's wave function by 2π.

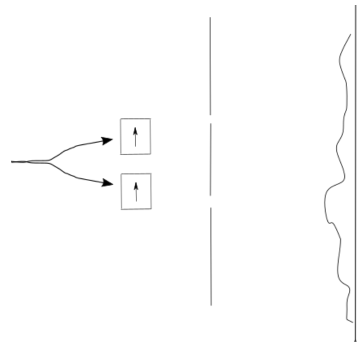

Figure 2. The electron wavefunction is split into two parts and placed into the two boxes. One of the boxes has wave function ψ_1 and the other has wave function ψ_2. The superposition is that the electron is in both boxes at the same time.