Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 6

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Abstract

This paper contains my notes on Lecture Six of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. This time we get going on quantum field theory.

1 Harmonic Oscillator Review

The harmonic oscillator is the central concept and tool of the subject as we shall learn it. For the moment, we'll concentrate on a single harmonic oscillator, especially on the operators of a^+ and a^-, the former operator raises the level of the state, and the latter decreases it. If there is a ground state $|\,0\,\rangle$, then

$$a^- \,|\,0\,\rangle = 0\,. \tag{1}$$

In other words, the $|\,0\,\rangle$ state is "annihilated" by the a^- operator. At some point, we'll refer to a^+ and a^- as "creation" and "annihilation" operators, respectively.

Recall that:

$$a^+ = \frac{(P + i\omega x)}{\sqrt{2\omega}}\,, \qquad a^- = \frac{(P - i\omega x)}{\sqrt{2\omega}}\,. \tag{2}$$

These two operators are hermitian conjugates of each other. So, with these symbolic simplifications, we can write

$$H = \omega a^+ a^-\,, \tag{3}$$

where we have ignored the constant term, which relates to the ground state energy.[1]

It is found to be convenient to give a name to a^+a^-, namely

$$N \equiv a^+ a^-\,. \tag{4}$$

So, let $|\,n\,\rangle$ be an eigenstate of N, with equation

$$N\,|\,n\,\rangle = n\,|\,n\,\rangle\,, \tag{5}$$

where n is an eigenvalue, or rather, an observable value.

Let's find the commutator of a^- and a^+:

$$[\,a^-, a^+\,] = 1\,. \tag{6}$$

[1] As always, the presence of $\hbar$ is ghostly, appearing and disappearing randomly.

Now, we undertake the problem of the many harmonic oscillator. A useful model of this coupled oscillation problem is to imagine the many vibrating modes of a single string, each with its own frequency.

So, switching to the multimode case, we need to label the operators, etc.

$$N_i \equiv a_i^+ a_i^- \,. \tag{7}$$

So, let $|\, n_i \,\rangle$ be an eigenstate of N_i, with equation

$$N_i \,|\, n_i \,\rangle = n_i \,|\, n_i \,\rangle \,. \tag{8}$$

Hence, the commutator of a_i^- and a_i^+:

$$[\, a_i^- , a_i^+ \,] = 1 \,. \tag{9}$$

But, since the operators of one subsystem are independent of the operators of any other, we need to write

$$[\, a_i^- , a_j^+ \,] = \delta_{ij} \,. \tag{10a}$$

Add to this:

$$[\, a_i^- , a_j^- \,] = 0 \,, \tag{10b}$$

$$[\, a_i^+ , a_j^+ \,] = 0 \,. \tag{10c}$$

Therefore, the Hamiltonian becomes

$$H = \sum_i \hbar\omega_i a_i^+ a_i^- = \sum_i \hbar\omega_i N_i \,. \tag{11}$$

Operator N_i will be called an "occupation number." We'll label the states by their occupation numbers, $|\, n_i \,\rangle$

We need a symbolism to represent multiple oscillators of a system.

$$|\, n_1 \, n_2 \, n_3 \; \ldots \rangle \,. \tag{12}$$

One model of this system would be an idealized violin string, which can oscillate in multiple modes (frequencies) at the same time. These are called "harmonics."

From a previous lecture, we have the result

$$a^- a^+ \,|\, n \,\rangle = (n+1) \,|\, n \,\rangle \;, \tag{13}$$

which we can use to calculate α_n in

$$a^+ \,|\, n \,\rangle = \alpha_n \,|\, n+1 \,\rangle \;, \tag{14}$$

which will give us the correct adjustment factor.

We begin with the requirement that the states be nornmalized according to

$$\langle\, n \mid m \,\rangle = \delta_{nm} \,. \tag{15}$$

Thus, on multiplying (14) through by its hermitian conjugate, we have

$$(\langle\, n \,|\, a^-)(a^+ \,|\, n \,\rangle) = \langle\, n \,|\, (a^- a^+) \,|\, n \,\rangle = (\langle\, n+1 \,|\, \alpha_n^*)(\alpha_n \,|\, n+1 \,\rangle) \,, \tag{16}$$

where

$$\langle\, n \,|\, a^- = (\, a^+ \,|\, n \,\rangle \,)^\dagger = (\alpha_n \,|\, n+1 \,\rangle \,)^\dagger = \langle\, n+1 \,|\, \alpha_n^* \,. \tag{17}$$

Employing (13),

$$(n+1)\,\langle\, n \mid n\,\rangle = |\,\alpha_n\,|^2\,\langle\, n+1 \mid n+1\,\rangle\ . \tag{18}$$

On applying our normalization constraint (15), we get

$$(n+1) = |\,\alpha_n\,|^2\,. \tag{19}$$

And, assuming that α_n is real (given that n is an integer), we get

$$\alpha_n = (n+1)^{1/2}\,. \tag{20}$$

Therefore, (14) becomes

$$a^{+}\,|\,n\,\rangle = (n+1)^{1/2}\,|\,n+1\,\rangle\ . \tag{21}$$

By similar reasoning, we get

$$a^{-}\,|\,n\,\rangle = n^{1/2}\,|\,n-1\,\rangle\ . \tag{22}$$

Naturally, when we hit our multistate vector by the creation operator a_i^{+}, we get

$$a_i^{+}\,|\,n_1\,n_2\,n_3\ \ldots\rangle = (n_i+1)^{1/2}\,|\,n_1\,n_2\ldots, n_i+1,\ \ldots\rangle\ . \tag{23}$$

Thus, it has the effect of changing only the ith occupation number and multiplying by the appropriate coefficient.

2 Quantum Fields, Intro

So, let's review some quantum mechanics, starting with the wave function $\psi(x)$, where x can represent a point in n-dimensional space. Now, in some respects $\psi(x)$ is a field, since it has values over the space of interest. However, it's not a function we can measure experimentally. We conclude that $\psi(x)$ is not an observable.

Now consider that our system is two particles, one at point x, the other at point y, then our representation of it in QM is:

$$\psi = \psi(x\,y)\,. \tag{24}$$

If we up the count to 15, say, then we write

$$\psi = \psi(x_1, \ldots,\, x_{15})\,. \tag{25}$$

But in both of these cases, the number of particles is fixed.

Now, to switch our view over to that of quantum field theory, we must make some major changes. First, we'll represent our quantum field by Ψ. Second, we insist that Ψ be an observable,[2] which we can enforce by restricting it to represent those things that already can be measured by experiment, at least as a start. By the way, the fact that Ψ is an observable implies that it is an operator on the space of states. Third, Ψ is a function of only one coordinate. Forth, Ψ describes systems of any number of particles, and can allow for a change of number of particles in a given situation. The old QM could not do that.

Note: At this point, I skipped some material that is more or less review.

So, referring back to $|\,n_1\,n_2\,n_3\ \ldots\rangle$, we regard these n's as indicating the number of particles in a given state. That is, n_i indicates the ith state with n_i particles in that state.

[2] Actually, we will need to manipulate this function to get a valid observable out of it.

Now we introduce the notion of the "vacuum" state, which is the state that is annihilated by the action of any annihilating operator. Logically, we should represent this state by the vector $|\,0\,0\,0\,\ldots\rangle$, hence, for arbitrary annihilation operator a_j^-

$$a_j^- \,|\,0\,0\,0\,\ldots\rangle = 0\,. \tag{26}$$

Anyway, it's clear that in that vacuum state, none of the states is occupied.

Now, each state has an associated energy, which we could label as E_i, but we won't. Instead, we'll express it as either ω_i or $\hbar\omega_i$. So, how should we represent the energy of a system in terms of creation-annihilation operators?

$$E = \sum_i n_i \omega_i\,, \tag{27}$$

which, if we switch over to the operator form of this equation, we get

$$E = \sum_i \omega_i N_i = \sum_i \omega_i a_i^\dagger a_i\,, \tag{28}$$

where we have made the standard notations swaps $a^+ \to a^\dagger$ and $a^- \to a$. (However, we will not make this switch at this time.) And, once again, we are ignoring the ground-state energy constant term.

Further comments on QFT:

QFT is a bookkeeping device to manage the multi-particles systems allowable by the theory. Anyway, for the time being, this is the way we will think of them.

Now, the space of vectors denoted by $|\,n_1\,n_2\ldots, n_i+1, \ldots\rangle$ was named after the Russian physicist V. A. Fock. The main point of which is to allow for a variable number of particles in the system.

Next, we come to the formal definition of the operator $\Psi(x)$[3]:

$$\Psi(x) = \sum_i a_i^- \psi_i(x)\,, \tag{29}$$

where the $\psi_i(x)$ are typically sines and cosines. (It seems that we have gone back to the old way of representing the annihilation operators.) If the ψ's represent momentum eigenstates of sines and cosines, then the a^- oerators would resemble the coefficients that occur in Fourier analysis. Note that these $\psi_i(x)$'s are functions defined at a point.

The hermitian conjugate of (29) is, of course,

$$\Psi^\dagger(x) = \sum_i a_i^+ \psi_i^*(x)\,. \tag{30}$$

Now, when $\Psi(x)$ hits the vacuum state, it annihilates it. But things are more interesting when $\Psi^\dagger(x)$ hits it. In that case, we get a new particle in a superposition of states.

To continue, we consider old QM of a single particle. It has a complete set of states given by $|\,x\,\rangle$. Therefore, we can construct an identity operator:

$$\sum_x |\,x\,\rangle\,\langle\,x\,| = 1\,. \tag{31}$$

[3]The great value of this definition will become obvious as we go along. What it allows us to do is to put our most important operator equations in their simplest possible forms.

We can also write down for the energy eigenstates:

$$\sum_i |\, i\,\rangle\, \langle\, i\,| = 1\,, \tag{32}$$

again, for a single particle, and each i corresponds to a $\psi_i(x)$. Obviously, then, applying this operator to a particle at point x, yields

$$\sum_i |\, i\,\rangle \langle\, i \mid x\,\rangle = |\, x\,\rangle\ , \tag{33}$$

where $\langle\, i \mid x\,\rangle$ is the wave function $\psi_i^*(x)$, hence, we get

$$\sum_i \psi_i^*(x)\, |\, i\,\rangle = |\, x\,\rangle\ . \tag{34}$$

Now, we get a bit fancy. We replace $|\, i\,\rangle$ by

$$|\, i\,\rangle = a_i^+\, |\, 0\,\rangle\ . \tag{35}$$

As a result, we get write (34) as

$$\sum_i \psi_i^*(x)\, a_i^+\, |\, 0\,\rangle = |\, x\,\rangle\ . \tag{36}$$

But, because of (30), we can write

$$\Psi^\dagger(x)\, |\, 0\,\rangle = |\, x\,\rangle\ . \tag{37}$$

Hence – drum roll, please – $\Psi^\dagger(x)$ is an operator that creates a particle (in the energy eigenstates) at point x! By the way, we refer to $\Psi^\dagger(x)$ as a "field operator." On the other hand, $\Psi(x)$ will delete a particle at point x if there's a particle already there.

Next, we generalize. To place a particle at x and a particle at y, we write

$$\Psi^\dagger(y)\Psi^\dagger(x)\, |\, 0\,\rangle = |\, y\, x\,\rangle\ . \tag{38}$$

But, since these creation operators commute, we also have that

$$\Psi^\dagger(x)\Psi^\dagger(y)\, |\, 0\,\rangle = |\, x\, y\,\rangle = |\, y\, x\,\rangle\ . \tag{39}$$

This works because our particles at this point are exclusively bosons. The upshot of this demonstration is that we have made a connection between particles and fields: Apply the fields to the ground state, and out comes a particle. Let there be bosons!