Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 7

P. Reany

December 14, 2022

Abstract

This paper contains my notes on Lecture Seven of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes.

1 Review

We begin this lecture with the probability that a particle will be found at point x as given by the old QM:

$$\psi^*(x)\psi(x)\,, \tag{1}$$

which is, of course, a probability density. And we know that if $\psi(x)$ is normalized, that

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x) = 1\,. \tag{2}$$

And as a conversion between bra-ket notation and wave functions, we have that

$$\langle\, x \mid \psi\,\rangle = \psi(x)\,. \tag{3}$$

What we want to do now is to introduce a complete set of basis vectors. In the case of a particle in a box potential, we get a full set of energy eigenstates $\psi_i(x)$, which are sine functions that are characterized by their number of nodes. Then,

$$\int_{-\infty}^{\infty} \psi_i^*(x)\psi_j(x) = \delta_{ij}\,. \tag{4}$$

We can also write down for the energy eigenstates:

$$\sum_i |\, i\,\rangle\,\langle\, i\,| = 1\,. \tag{5}$$

So, by multiplying on the right by $|\,x\,\rangle$ and on the left by $\langle\, y\,|$, we get

$$\sum_i \langle\, y \mid i\,\rangle \langle\, i \mid x\,\rangle = \langle\, y \mid x\,\rangle\;, \tag{6}$$

which simplifies to

$$\sum_i \psi_i(y)\psi_i^*(x) = \delta(x-y)\,. \tag{7}$$

All we have asked of $\psi_i(x)$ is that they are a complete set of orthonormal basis vectors.

For a multi-particle system, we have already seen the notation

$$|\, n_1\, n_2\, n_3\, \ldots \rangle \,, \tag{8}$$

where the basis of states are the n_i's and they are called the "occupation numbers." In order to raise or lower these number, we associate to each i an "oscillator." These are the by now familiar creation/annihilation operators a_i^+ and a_i^-.

At this point, we make the conversion to the alternative convention:

$$a_i^+ \longrightarrow a_i^\dagger \quad \text{and} \quad a_i^- \longrightarrow a_i \,. \tag{9}$$

Next, we come to the formal definition of the operator $\Psi(x)$[1]:

$$\Psi(x) = \sum_i a_i \psi_i(x) \,, \tag{10}$$

where the $\psi_i(x)$ are typically sines and cosines. The hermitian conjugate of (10) is, of course,

$$\Psi^\dagger(x) = \sum_i a_i^\dagger \psi_i^*(x) \,. \tag{11}$$

Now we introduce the notion of the "vacuum" state, which is the state that is annihilated by the action of any annihilating operator. Logically, we should represent this state by the vector $|\,0\,0\,0\,\ldots\rangle$,

$$|\,\text{vacuum}\,\rangle = |\,0\,\rangle = |\,0\,0\,0\,\ldots\rangle \,. \tag{12}$$

Hence, for arbitrary annihilation operator a_j^-:

$$a_j^- \,|\,0\,0\,0\,\ldots\rangle = 0 \,. \tag{13}$$

To continue, we consider old QM of a single particle. It has a complete set of states given by $|\,i\,\rangle$. We can also write down for the energy eigenstates:[2]

$$\sum_i |\,i\,\rangle \langle\, i\,| = 1 \,, \tag{14}$$

again, for a single particle, and each i corresponds to a $\psi_i(x)$. Obviously, then, applying this operator to a particle at point x, yields

$$|\,x\,\rangle = \sum_i |\,i\,\rangle \langle\, i \mid x\,\rangle \,, \tag{15}$$

where $\langle\, i \mid x\,\rangle$ is the wave function $\psi_i^*(x)$, hence, we get

$$|\,x\,\rangle = \sum_i \psi_i^*(x)\,|\,i\,\rangle \,. \tag{16}$$

Now, we get a bit fancy. We replace $|\,i\,\rangle$ by

$$|\,i\,\rangle = a_i^\dagger\,|\,0\,\rangle \,. \tag{17}$$

[1]The great value of this definition will become obvious as we go along. What it allows us to do is to put our most important operator equations in their simplest possible forms.

[2]This identity has the special name "the resolution of the identity."

As a result, we get write (16) as

$$| \, x \, \rangle = \sum_i \psi_i^*(x) \, a_i^\dagger \, | \, 0 \, \rangle \, . \tag{18}$$

But, because of (11), we can write

$$| \, x \, \rangle = \Psi^\dagger(x) \, | \, 0 \, \rangle \, . \tag{19}$$

Hence, $\Psi^\dagger(x)$ is an operator that creates a particle (in the energy eigenstates) at point x! By the way, we refer to $\Psi^\dagger(x)$ as a "field operator." On the other hand, $\Psi(x)$ will delete a particle at point x if there's a particle already there.

2 New Material

Consider the integral

$$\int dx \, \Psi^\dagger(x)\Psi(x) \, . \tag{20}$$

What can we say about it? One thing we can say is that it is hermitian because $\Psi^\dagger(x)\Psi(x)$ is hermitian. Now,

$$\begin{aligned}
\int dx \, \Psi^\dagger(x)\Psi(x) &= \int dx \sum_{i,j} a_i^\dagger \psi_i^*(x) \, a_j \psi_j(x) \\
&= \sum_{i,j} a_i^\dagger \, a_j \, \delta_{ij} \quad \text{(using (4))} \\
&= \sum_i a_i^\dagger \, a_i \\
&= \sum_i N_i \, . \qquad (21)
\end{aligned}$$

Of course, we recognize this sum as the total number of particles in the system. But, does this integral blow up? It might, because there might be an infinite number of energy levels. On the other hand, if the total engery of the system is capped (which is the physical situation), then that finite energy cannot be divided into an infinite number of particles, as least so long as each particle as some energy. Anyway, it makes sense to regard $\Psi^\dagger(x)\Psi(x)$ as the density of particles.

Next, we wish to examine an approximation in which the density of particles is small enough to warrant making the assumption that they will not interact appreciably with each other. Hence, the total energy of the system is the simple sum of the individual particle energies. (The particles are considered as 'free'.) Therefore,

$$E = \sum_i N_i \omega_i = \sum_i a_i^\dagger \, a_i \omega_i \, , \tag{22}$$

where we have left off the $\hbar$ factor.

Now, we investigate the action of the Hamiltonian on ψ_i:

$$H\psi_i = \omega_i \psi_i \, . \tag{23}$$

On expanding for H, we get

$$\Big(\frac{P^2}{2m} + V(x) \Big) \psi_i = \omega_i \psi_i \, . \tag{24}$$

But, since $P = -i\partial/\partial x \longrightarrow -i\nabla$. Then,

$$\Big(\frac{-\nabla^2}{2m} + V(x) \Big) \psi_i = \omega_i \psi_i \, . \tag{25}$$

Note that the expectation value of the energy can be written as

$$\int \psi^*(x)\Big(\frac{-\nabla^2}{2m} + V(x)\Big)\psi(x) = \langle\, \psi(x)\,|\, H \,|\; \psi(x)\rangle \; . \tag{26}$$

Now let's upgrade the ψ in (26) to Ψ:

$$\int \Psi^\dagger(x)\Big(-\frac{\nabla^2}{2m} + V(x)\Big)\Psi(x)\,, \tag{27}$$

and so,

$$\begin{aligned}
\int \Psi^\dagger(x)\Big(-\frac{\nabla^2}{2m} + V(x)\Big)\Psi(x) &= \int dx \sum_{i,j} a_i^\dagger \psi_i^*(x)\Big(-\frac{\nabla^2}{2m} + V(x)\Big)a_j\psi_j(x) \\
&= \sum_{i,j} a_i^\dagger a_j \int dx\, \psi_i^*(x)\Big(-\frac{\nabla^2}{2m} + V(x)\Big)\psi_j(x) \\
&= \sum_{i,j} a_i^\dagger a_j \int dx\, \psi_i^*(x)\,\omega_j\psi_j(x) \\
&= \sum_{i,j} a_i^\dagger\, a_j\omega_j\delta_{ij} \quad \text{(using (4))} \\
&= \sum_i \omega_i a_i^\dagger a_i \\
&= \sum_i \omega_i N_i \, .
\end{aligned} \tag{28}$$