Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 8

P. Reany

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Abstract

This paper contains my notes on Lecture Eight of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes.

1 Particle sharing two sides of a box

This time we consider the case of a particle sharing two sides of a box, with a finite potential separating the two sides. In this case, the particle is able to 'inhabit' both sides of the box — one waveform being symmetric, and the other being antisymmetric. See Fig. 1.

The energy of the symmetric form across the center boundary is slighly more than if the particle's waveform were restricted to one side or the other. On the other hand, for the antisymmetric case, the energy of the antisymmetric form across the center boundary is slighly less than if the particle's waveform were restricted to one side or the other.

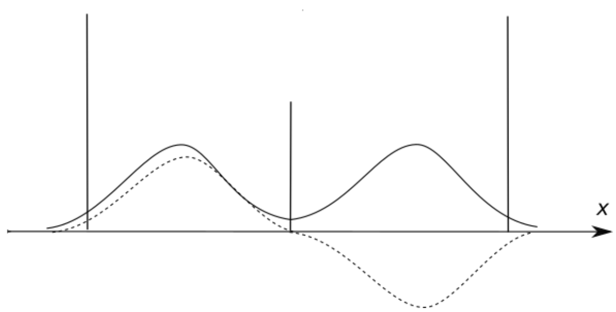

Figure 1. The solid waveform represents the symmetric eigenfunction, which has no nodes. The dotted waveform represents the anti-symmetric eigenfunction, which has one node.

The particle being placed in one side will have an evolving wavefunction to admit to having the particle appear in both sides:

$$\frac{\psi_L + \psi_R}{\sqrt{2}} \sim E_1 - \epsilon\,, \tag{1}$$

$$\frac{\psi_L - \psi_R}{\sqrt{2}} \sim E_1 + \epsilon\,, \tag{2}$$

where E_1 is the energy associated with starting in one side or the other.

To prepare these waveforms to be used in the time-dependent Schrödinger equation, we have that

$$\psi_+(x) = \frac{\psi_L + \psi_R}{\sqrt{2}} e^{(E_1-\epsilon)t} \,, \tag{3}$$

$$\psi_-(x) = \frac{\psi_L - \psi_R}{\sqrt{2}} e^{(E_1+\epsilon)t} \,. \tag{4}$$

Now, how does the wavefunction evolve if the particle starts off in the left chamber of the box? So, at $t = 0$ we have just ϕ_L.

$$\phi_L = \frac{\psi_+ + \psi_-}{\sqrt{2}} \,. \tag{5}$$

Then, after time t:

$$\begin{aligned} \phi_L(t) &= \frac{\psi_+ e^{(E_1-\epsilon)t} + \psi_- e^{(E_1+\epsilon)t}}{\sqrt{2}} \\ &= e^{E_1 t} \frac{\psi_+ e^{-\epsilon t} + \psi_- e^{\epsilon t}}{\sqrt{2}} \,. \end{aligned} \tag{6}$$

Our question now is: How long will it take until the two phases have the opposite signs? That will occur when $e^{-\epsilon t} = -e^{\epsilon t}$, or, when $t = 2\epsilon\pi$. At this time, the particle will likely have materialized on the other side. Therefore, the phenomonon of mixing goes together with the phenomenon of oscillations.

Now, an electron neutrino is produced in a state that is similar to being in a pure single state and then oscillating between that state and its complementary state of a muon neutrino. The theory says that in time the electron neutrino will evolve into a muon neutrino.

For an analogous situation, consider a spinning particle that is prepared in the up direction, namely $|\!\uparrow\rangle$. If we admit this particle into the space with a magnetic field pointing in the $+x$ direction, the particle spin direction will precess around the $+x$ axis. We can write the left spin direction $|\!\leftarrow\rangle$ as a linear combination of $|\!\uparrow\rangle$ and $|\!\downarrow\rangle$: At this point, we make the conversion to the alternative convention:

$$|\!\leftarrow\rangle = \frac{|\!\uparrow\rangle + |\!\downarrow\rangle}{\sqrt{2}} \,. \tag{7}$$

In this precession, an originally up electron will act like a down electron.

Note: I have skipped a section concerning the electric charge distribution of the electron.

2 Field operators and Fourier transforms

In the old QM, we had for the probability of finding the particle at position x:

$$P(x) = \psi^*(x)\psi(x) \,. \tag{8}$$

If we want to Fourier transform $\psi(x)$ into the momentum basis to get $\widetilde{\psi}(p)$, with Fourier transforms, then we use:

$$\widetilde{\psi}(p) = \int \frac{dx}{\sqrt{2\pi}} \psi(x) e^{-ipx} \,, \tag{9}$$

and

$$\psi(x) = \int \frac{dp}{\sqrt{2\pi}} \widetilde{\psi}(p) e^{ipx} \,. \tag{10}$$

Then the probability in this basis is

$$P(p) = \widetilde{\psi}^*(p)\widetilde{\psi}(p)\,. \tag{11}$$

Once again, the field operator for annihilation is:[1]

$$\Psi(x) = \sum_i a_i \psi_i(x)\,, \tag{13}$$

Question: What happens if we are dealing with a free particle on an infinite axis, whose energy eigenstates are continuous? Well, we have moved from the discrete case, indexed by i, to the continuous states, which require us to integrate instead. Thus,

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int dp\, a(p) e^{ipx}\,, \tag{14}$$

where $a(p)$ removes a particle of momentum p.

Next, the creation field operator:

$$\Psi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int dp\, a^\dagger(p) e^{-ipx}\,. \tag{15}$$

Going back a few lectures, we had that

$$[\, a_i,\, a_j^\dagger \,] = \delta_{ij}\,. \tag{16}$$

By analogy,

$$[\, \Psi^\dagger(x),\, \Psi(y) \,] = \delta(x-y)\,. \tag{17}$$

(To be consistent, I would think that this commutator should be taken in the reverse order.)

That's all for this lecture — next time we consider fermion operators.

[1] Note that I am still using the updated creation/annihilation operator symbols

$$a_i^+ \longrightarrow a_i^\dagger \qquad \text{and} \qquad a_i^- \longrightarrow a_i\,. \tag{12}$$