Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 9

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Abstract

This paper contains my notes on Lecture Nine of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes.

1 Hamiltonian

The Hamiltonian in the field operators is given as

$$\begin{aligned} H &= \int dx\, \Psi^\dagger(x)\Big(-\frac{\nabla^2}{2m} + V(x)\Big)\Psi(x) \\ &= \int dx\, [\, \Psi^\dagger(x)(-\frac{\nabla^2}{2m})\Psi(x) + V(x)\Psi^\dagger(x)\Psi(x)\,]\,, \end{aligned} \tag{1}$$

where $-\nabla^2/2m$ is the kinetic energy term and $V(x)$ is the potential energy term. The term $V(x)\Psi^\dagger(x)\Psi(x)$ counts the number of particles at point x, assigning each of them the potential energy $V(x)$.

Now, let's reconfigure (1) to allow for only a constant potential term (which applies no force on the system):

$$H = \int dx\, [\, \Psi^\dagger(x)(-\frac{\nabla^2}{2m})\Psi(x) + \Psi^\dagger(x)\Psi(x)\,]\,. \tag{2}$$

Question: How does the second term now relate to the energy? Ans: It merely counts the number of particles – at a certain energy. But this is not the kinetic energy, as that is contained in the first term. Thus, we'll assign it to the rest mass of each particle, as such:

$$H = \int dx\, [\, \Psi^\dagger(x)(-\frac{\nabla^2}{2m})\Psi(x) + mc^2\Psi^\dagger(x)\Psi(x)\,]\,. \tag{3}$$

The role of the Hamiltonian is to update the state of the system. Let's indicate the state of the system at time t as $|\,\varphi(t)\,\rangle$. For small ϵ,

$$|\,\varphi(t+\epsilon)\,\rangle \approx (1 - i\epsilon H)\,|\,\varphi(t)\,\rangle = |\,\varphi(t)\,\rangle - i\epsilon H\,|\,\varphi(t)\,\rangle\ . \tag{4}$$

We can think of the function of the Hamiltonian as updating the wavefunction continuously in time. But what happens when the Hamiltonian acts on a definite momentum state? In the special case of momentum conservation, of course, it won't change it.

Let's check this claim. For starters, let's use a Fourier transform to rewrite (3) in the momentum basis.[1] For the annihilation operator, we have

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int dp\, \widetilde{\Psi}(p) e^{ipx}\,, \tag{5}$$

and for the creation operator, we have[2]

$$\Psi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int dq\, \widetilde{\Psi}^\dagger(q) e^{-iqx}\,. \tag{6}$$

One step we need to take is to rewrite $mc^2\Psi^\dagger(x)\Psi(x)$ in terms of momentum variables.

$$\begin{aligned} H &= \int dx\, mc^2 \Psi^\dagger(x)\Psi(x) \\ &= \frac{1}{2\pi}\int dx\, mc^2 \int dq\, \widetilde{\Psi}^\dagger(q) e^{-iqx} \int dp\, \widetilde{\Psi}(p) e^{ipx} \\ &= \frac{1}{2\pi}\int dx \int dq \int dp\, \widetilde{\Psi}^\dagger(q)\widetilde{\Psi}(p) mc^2 e^{i(p-q)x} \\ &= \int dq \int dp\, \widetilde{\Psi}^\dagger(q)\widetilde{\Psi}(p) mc^2 \delta(p-q) \\ &= \int dp\, \widetilde{\Psi}^\dagger(p)\widetilde{\Psi}(p)\, mc^2\,. \end{aligned} \tag{7}$$

Thus, we found that for the integral to work, p and q must be equal. As a result, $\widetilde{\Psi}(p)$ will remove a particle of momentum p, but $\widetilde{\Psi}^\dagger(p)$ will bring it right back, leaving the number of particles at momentum p conserved.

Suppose now that we generalize the mass term in Equation (3) to

$$mc^2(\Psi^\dagger(x)_A\Psi(x)_C + \Psi^\dagger(x)_B\Psi(x)_D)\,. \tag{8}$$

Then we'd get in the integral (as an example of a more complicated system)

$$\int \cdots \int dx\, dp_A dp_B dq_C dq_D mc^2 (\widetilde{\Psi}^\dagger(p)_A \widetilde{\Psi}(q)_C + \widetilde{\Psi}^\dagger(p)_B \widetilde{\Psi}(q)_D) e^{i(p_A+p_B)x} e^{-i(q_C+q_D)x}\,. \tag{9}$$

p's go with the annihilation operators, and q's go with the creation operators. Integrating over the x, we get

$$\int \cdots \int dp_A dp_B dq_C dq_D mc^2 (\widetilde{\Psi}^\dagger(p)_A \widetilde{\Psi}(q)_C + \widetilde{\Psi}^\dagger(p)_B \widetilde{\Psi}(q)_D) \delta(p_A + p_B - q_C - q_D)\,. \tag{10}$$

We will interpret this delta function as meaning that the sum of the p's must equal the sum of the q's. In other words, the sum of the momentum that is removed is equal to the sum of the momentum that is put back in.

Now it's time to look at the kinetic energy term in the Hamiltonian.

$$-\nabla^2\Psi^\dagger(x) = -\nabla^2 \frac{1}{\sqrt{2\pi}} \int dq\, a^\dagger(q) e^{-iqx} = \frac{1}{\sqrt{2\pi}} \int dq\, a^\dagger(q) q^2 e^{-iqx}\,. \tag{11}$$

[1] We will write this equation as if p is 1-dimensional, but it works for higher dimensions as well.
[2] Note: there is a factor of $1/\sqrt{2\pi}$ for each dimension of space.

Then, with momentum conserved, the total kinetic energy is given by

$$\int dp\, \widetilde{\Psi}^\dagger(p)\widetilde{\Psi}(p)\frac{p^2}{2m}\,. \tag{12}$$

Now, since $\int dp\, \widetilde{\Psi}^\dagger(p)\widetilde{\Psi}(p)$ counts the number of particles, then $\int dp\, \widetilde{\Psi}^\dagger(p)\widetilde{\Psi}(p)p^2$ adds up the total amount of kinetic energy by attaching p^2 to each particle.

Let's suppose this time that we have to kinds of particles. We'll call one of them an electron and the other a proton. I point out that these are merely names attached to bosons, not the actual fermion version of these particles.

$$H = \int dx\, [\,\Psi^\dagger(x)(-\frac{d^2}{dx^2})\Psi(x)\,] = \int dx\, [\,\Psi^\dagger(x)(-\frac{d}{dx}\frac{d}{dx})\Psi(x)\,]\,. \tag{13}$$

Next, we perform an integration by parts.

$$H = \int dx\, \frac{d}{dx}\Psi^\dagger(x)\,\frac{d}{dx}\Psi(x)\,. \tag{14}$$

By imagining the expression $\frac{d}{dx}\Psi^\dagger(x)\,\frac{d}{dx}\Psi(x)$ as the square of a derivative of a field, we get a particular form of the generic term in a Hamiltonian in a field theory.

$$\begin{aligned} H = &\int dx\, [\,\Psi_e^\dagger(x)(-\frac{\nabla^2}{2m})\Psi_e(x) + \Psi_p^\dagger(x)(-\frac{\nabla^2}{2m})\Psi_p(x)\,] \\ &+ \int dx\, \widetilde{\Psi}_e^\dagger(x)\widetilde{\Psi}_p^\dagger(x)\widetilde{\Psi}_e(x)\widetilde{\Psi}_p(x)\,, \end{aligned} \tag{15}$$

where the first term is the KE of all the electrons and the second terms is the KE of all the protons. The third term will only act to remove an electron-proton pair it they are at point x at the same time. If this interaction does simultaneously occur, it will then be followed up by the creation of an electron-proton pair, the net effect of which is to scatter the incoming pair to an outgoing pair. And, of course, the overall momentum of the particles is conserved in this reaction.

The third term is therefore appropriate to put into a Hamiltonian for the interaction of two particle which appear to interact only by very short-range interaction potentials.

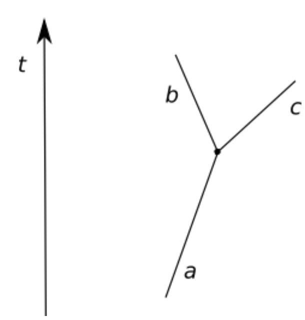

Figure 1. Depicted is the decay of particle a that disappears at the interaction point, and then particles b and c appear. Our job is to correlate this interaction inside a Hamiltonian.

The following equation attaches to the interaction presented in spacetime diagram Fig. 1.

$$H = g\int dx\, \Psi_b^\dagger(x)\Psi_c^\dagger(x)\Psi_a(x)\,. \tag{16}$$

As before, we could adjust this equation to show that momentum is conserved. We have added a factor of g that indicates how likely it is (or isn't) that a particular interaction will occur. It is referred to as the 'coupling constant'.

However, (16) is not in the correct form, because H is hermitian, but (16) isn't. Let's fix this. The fix is easy:

$$H = g \int dx\, \Psi_b^\dagger(x)\Psi_c^\dagger(x)\Psi_a(x) + g \int dx\, \Psi_a^\dagger(x)\Psi_c(x)\Psi_b(x)\,. \tag{17}$$

This second term has the associated graphic:

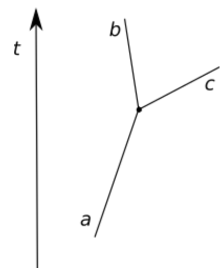

Figure 2. Depicted is the interaction at point x. In comes particle a, which is then annihilated, and out goes particles b and c.

So, now we have the basic rules of a simple quantum field theory.

2 Hamiltonian terms of second-order

At this point, we return to Eq. (4), only this time we will include a term of one more order:

$$|\,\varphi(t+\epsilon)\,\rangle \approx |\,\varphi(t)\,\rangle - i\epsilon H\,|\,\varphi(t)\,\rangle - \frac{\epsilon^2}{2} H^2\,|\,\varphi(t)\,\rangle\ . \tag{18}$$

How shall we deal with the H^2 term? Instead of squaring it out, we'll use pictures to help us out.

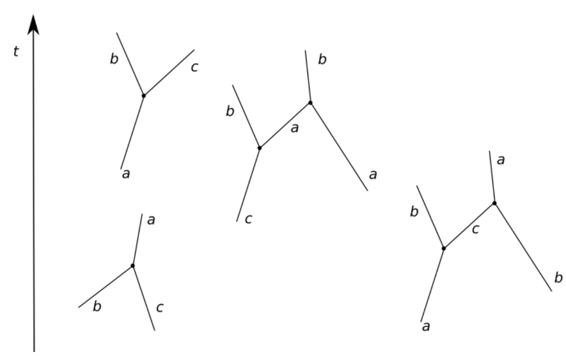

Figure 3. The more complicated figure in the figure is second-order in the Hamiltonian. a and b have scattered with each other with the exchange of a c. As a second-order effect, its likelihood of occurrence would go as g^2.

In 'squaring' the hamiltonian, we get the term

$$H = \Psi_c^\dagger(x)\Psi_b^\dagger(x)\Psi_a(x)\,\Psi_a^\dagger(x)\Psi_a(x)\Psi_c(x)\,. \tag{19}$$

The effect of this term is to conjoin b and c to make an a, and then to annihilate a to remake b and c, effectively allowing b and c to scatter off each other. So, we find that the Hamiltonian and all it powers update the state.

Next, we consider electrons interacting with each other and with the electromagnetic field.

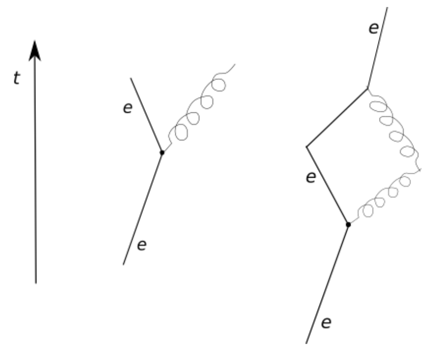

Figure 4. Depicted on the left is an electron emitting a photon. The operator to describe this interaction is $A\Psi_e^\dagger(x)\Psi_e(x)$, where A is the photon creation operator. Depicted on the right is a figure that suggests a second-order effect.

The second-order effect should be interpreted more as an adjustment to the first-order effects, suggesting an electron-photon pair (a superposition), rather than a whole new process.

3 The Dirac Equation

Following Dirac's motivation, we wish top place spae and time on an equal footing to arrive at a special-relativistic replacement of the Schrödinger equation. We begin with the classical energy equation:

$$E^2 = P^2 + m^2 \qquad (c = 1)\,. \tag{20}$$

One could try the obvious substitution:

$$E \longrightarrow i\partial/\partial t\,, \tag{21}$$

and then (20) becomes the Klein-Gordon equation:

$$-\frac{\partial^2\phi}{\partial t^2} = -\frac{\partial^2\phi}{\partial x^2} + m^2\phi\,. \tag{22}$$

But Dirac wanted an equation operator in the first power of E, not the second power. Let's try an ansatz, then:

$$i\frac{\partial\psi}{\partial t} = \sqrt{P^2 + m^2}\,\psi\,. \tag{23}$$

Well, let's hope we can do better than that. For our next ansatz, let's try to find the correct equation for a massless particle. This particle must move at the speed of light, and has momentum $P = E$. In particular, this particle moves to the right.

$$i\frac{\partial\psi}{\partial t} = -i\frac{\partial\psi}{\partial x}\,. \tag{24}$$

Or, on rearranging,

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} = 0\,. \tag{25}$$

The solutions to this equation are of the form

$$\psi = f(x - t)\,, \tag{26}$$

which describes a rigid wave moving to the right.

Now, there are two problems with this solution that we have to fix. The first is that we are allowed negative momenta, but not negative energies, which means that it's then possible to fill the world with negative energy particles without a boundary limit. The second problem is that it only has particles that can move to the right.

To deal with the asymmetry of motion problem, Dirac suggested that we hypothesize the existence of two 'species' of electrons: one kind that moves to the right ψ_1, and one that moves to the left ψ_2. This gives us the pair of equations:

$$\frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_1}{\partial x} = 0\,, \tag{27}$$

$$\frac{\partial \psi_2}{\partial t} - \frac{\partial \psi_2}{\partial x} = 0\,. \tag{28}$$

With the idea that our coupled wavefunction involve the notion of a sign duality ± 1, let's express the last set of coupled equations in matrix form, such that we can highlight this notion of a $(+1)$ eigenstate and a (-1) eigenstate. Therefore, the left-right observable for this discrete symmetry is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,. \tag{29}$$

Dirac named this 2×2 matrix α. So, we end up with the two equations

$$H = \alpha P\,, \tag{30}$$

and thus

$$i\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,. \tag{31}$$

We can now simplify this equation by writing ψ as the two- component column vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,. \tag{32}$$

Then,

$$i\frac{\partial \psi}{\partial t} = -i\alpha\frac{\partial \psi}{\partial x}\,. \tag{33}$$

The last problem that Dirac had was to manipulate the Hamiltonian someway so as to introduce mass, otherwise this equation would be stuck on massless particles. So, he tried

$$H = \alpha P + \beta m\,, \tag{34}$$

where m is the rest mass, but he still had to work with the two-component column vectors, and 2×2 matrix operators. So, β is a 2×2 matrix to be consistent with the fact that α is a 2×2 matrix.

Now, we still have the constraint equation that (34) has to agree with

$$E^2 = P^2 + m^2\,. \tag{35}$$

On squaring (34), we get

$$E^2 = (\alpha P + \beta m)(\alpha P + \beta m) = \alpha^2 P^2 + mP(\alpha\beta + \beta\alpha) + \beta^2 m^2 \,. \tag{36}$$

On comparing these last two equations, we must have that

$$1 = \alpha^2 = \beta^2 \,, \tag{37a}$$

$$0 = \alpha\beta + \beta\alpha \,. \tag{37b}$$

Now, our 2×2 α matrix clearly squares to the 2×2 identity matrix, and a suitable Pauli matrix for β that anticommutes with α would be

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{38}$$

Thus, we rewrite (31) as

$$i\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{39}$$

This gives us the pair of coupled equations:

$$i\frac{\partial \psi_1}{\partial t} + i\frac{\partial \psi_1}{\partial x} = m\psi_2 \,, \tag{40}$$

$$i\frac{\partial \psi_2}{\partial t} - i\frac{\partial \psi_2}{\partial x} = m\psi_1 \,. \tag{41}$$

Now that the equations are coupled, what speed does the particle move at? Well, less than the speed of light because, by design, it was set up to conform to (20).

Now, what at first seemed silly about this moving to the right needing to be countered by also moving to the left, turned out to make sense: for, when viewed from the special relativistic point of view, a mass particle can move either to the left or right, depending on how one boosts one's frame, i.e., performs a Lorentz transformation.