Group velocity and the stationary phase approximation

P. Reany

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Abstract

This presentation uses my read-along notes on the lecture from Barton Zwiebach: MIT 8.04, Spring 2013: Group velocity and stationary phase approximation. The fault for any inaccuracies in this presentation is strictly my own.

1 Some formulas

Frequency of matter waves, etc:

$$p = \hbar k, \quad \omega = E/\hbar \,. \tag{1}$$

Wave phase: $kx - \omega t$:

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{E}{p} = \frac{\frac{1}{2}mv^2}{mv} = \tfrac{1}{2}v\,, \tag{2}$$

$$v_{\text{group}} = \left.\frac{d\omega}{dk}\right|_k = \frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p}{m} = v\,. \tag{3}$$

2 Group velocity

We start off with the basic assumption that

$$\omega = \omega(k)\,. \tag{4}$$

Next, we define group velocity as the velocity of a wavepacket constructed by the superposition of waves. Hence,

$$\psi(x,t) = \int dk\, \Phi(k)\, e^{i(kx - \omega(k)t)}\,. \tag{5}$$

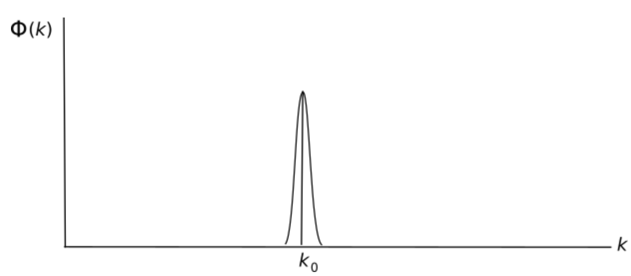

Figure 1. For this wavepacket, the wave numbers are closely spread around $k = k_0$.

So, how should we characterize the wavepacket movement?

To analyze this question informally, we will use the *Principle of Stationary Phase.*

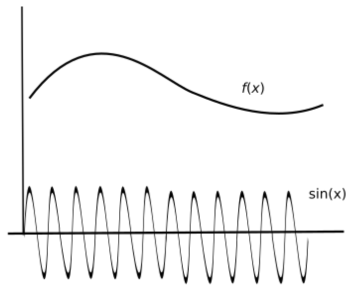

Figure 2. An integrand with a slowly varying $f(x)$ but a rapidly varying sine function will average out to near zero. To get a non-zero result, we need to couple the slowly varying $f(x)$ with a slowly varying sin(x). Hence, a stationary phase approximation.

Returning to (5), $\Phi(k)$ in nonzero only around k_0, then only when the phase stops varying much do we get a significant contribution to the integral. Setting

$$\phi(k) = e^{i(kx-\omega(k)t)}\,, \tag{6}$$

we realize this condition by applying the differential-equation critical point by

$$\left.\frac{d\phi(k)}{dk}\right|_{k=k_0} = 0\,, \tag{7}$$

or

$$x - \left.\frac{d\omega(k)}{dk}\right|_{k=k_0} t = 0\,. \tag{8}$$

Thus,

$$x = \left.\frac{d\omega(k)}{dk}\right|_{k=k_0} t = v_{\mathrm{group}}\,t\,. \tag{9}$$