The Heisenberg Picture for Quantum Mechanics

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Abstract

This paper contains my read-a-long notes on how to develop the quantum mechanical equations for the Heisenberg Picture. These notes come from the lecture from Barton Zwiebach[1]. The fault for any inaccuracies in this presentation is strictly my own.

1 Introduction

With the Heisenberg picture for quantum mechanics, we'll see how the Schrödinger oscilator acquires time dependence. And we'll find a greater connection between classical mechanics and quantum mechanics. So we begin.

$$|\,\psi, t\,\rangle = U(t,t_0)\,|\,\psi, t_0\,\rangle\,. \tag{1}$$

implies the Schrödinger equation with Hamiltonian

$$H(t) = i\hbar\left(\frac{\partial}{\partial t}U(t,t_0)\right)U^\dagger(t,t_0)\,. \tag{2}$$

Our goal is to find $U(t,t_0)$ given $H(t)$, which is often made-to-order.

We can multiply through on the right on the last equation by $U(t,t_0)$ and reverse sides, to get

$$i\hbar\frac{d}{dt}U(t,t_0) = H(t)U(t,t_0)\,. \tag{3}$$

And from this we get the Schrödinger equation

$$i\hbar\frac{d}{dt}\big(U(t,t_0))\,|\,\psi, t_0\,\rangle\big) = H(t)\big(U(t,t_0))\,|\,\psi, t_0\,\rangle\big)\,. \tag{4}$$

Now, let's go to cases!

Case 1) H is time-independent.

$$i\hbar\frac{dU}{dt} = HU\,. \tag{5}$$

We'll try the ansatz

$$U = e^{-iHt/\hbar}U_0\,, \tag{6}$$

[1]MIT 8.05 Quantum Physics II, Fall 2013, 13. Quantum Dynamics (con't) Heisenberg Picture.

where U_0 is a constant matrix. On substituting into (5) we get

$$i\hbar \frac{dU}{dt} = i\hbar \frac{dU}{dt} = H(t)U \,. \tag{7}$$

So,

$$i\hbar \frac{dU}{dt} = i\hbar \frac{-iH}{\hbar} e^{-iHt/\hbar} U_0 = HU \,, \tag{8}$$

and it works.

Now,

$$U(t, t_0) = e^{-iHt/\hbar} U_0 \,. \tag{9}$$

At $t = t_0$, $U(t_0, t_0) = \mathbb{1}$, so this last equation becomes

$$\mathbb{1} = e^{-iHt_0/\hbar} U_0 \,. \tag{10}$$

So, on solving for U_0, we get

$$U_0 = e^{iHt_0/\hbar} \,. \tag{11}$$

Therefore, we get

$$U(t, t_0) = e^{-iH(t-t_0)/\hbar} \,, \tag{12}$$

again, for the time-independent case. And we have that

$$e^{\alpha H} \,|\, \psi_n \,\rangle = e^{\alpha E_n} \,|\, \psi_n \,\rangle \,, \tag{13a}$$

$$\text{if } H \,|\, \psi_n \,\rangle = E_n \,|\, \psi_n \,\rangle \,. \tag{13b}$$

Case 2) H has a little time dependence, with

$$[\, \widehat{H}(t_1), \widehat{H}(t_2) \,] = 0 \quad \text{for all } t_1, t_2 \,. \tag{14}$$

As an example, consider a magnetic field whose field lines are collinear in a region, but whose strength along a give line is allowed to vary.

$$H = -\gamma \widehat{B}(t) \cdot \widehat{S} \,. \tag{15}$$

So, if we have that

$$H = -\gamma B_z(t) \cdot \widehat{S}_z \,, \tag{16}$$

then $H(t_1)$ commutes with $H(t_2)$.

Let's try the ansatz

$$U(t) = \exp\,[\, -\frac{i}{\hbar} \int_{t_0}^{t} H(t')\, dt' \,] \,, \tag{17}$$

Let's define

$$-\frac{i}{\hbar} \int_{t_0}^{t} H(t')\, dt' \equiv R(t) \,. \tag{18}$$

Then

$$\dot{R} = -\frac{i}{\hbar} H(t) \,. \tag{19}$$

So,

$$U = e^{R} \,. \tag{20}$$

Therefore,

$$\begin{aligned}\frac{dU}{dt} &= \frac{d}{dt}\big(1 + R + \frac{1}{2}RR + \frac{1}{3!}RRR + \cdots\big) \\ &= \dot{R} + \tfrac{1}{2}(\dot{R}R + R\dot{R}) + \frac{1}{3!}(\dot{R}RR + R\dot{R}R + RR\dot{R}) + \cdots \\ &= \dot{R}e^{R}\,, \end{aligned} \tag{21}$$

where we used that R and $\dot{R}$ commute.

So,

$$\frac{dU}{dt} = \dot{R}e^{R} = -\frac{i}{\hbar}H(t)U\,, \tag{22}$$

which is the same as (3).

Case 3) $H(t)$ is general. We can at least write down something that makes sense.

$$\begin{aligned} U(t,t_0) &= \mathbb{T}\,\exp\,[-\frac{i}{\hbar}\int_{t_0}^{t}H(t')dt'\,] \\ &= 1 + \frac{-i}{\hbar}\int_{t_0}^{t}H(t_1)dt_1 + \tfrac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int_{t_0}^{t}H(t')dt'\int_{t_0}^{t}H(t'')dt'' + \cdots\,, \end{aligned} \tag{23}$$

where the $\mathbb{T}$ indicates a time-ordered exponential.

$$\begin{aligned} U(t,t_0) &= \mathbb{T}\,\exp\,[-\frac{i}{\hbar}\int_{t_0}^{t}H(t')dt'\,] \\ &= 1 + \frac{-i}{\hbar}\int_{t_0}^{t}H(t_1)dt_1 \\ &\quad + \left(\frac{-i}{\hbar}\right)^2\int_{t_0}^{t}H(t_1)dt_1\int_{t_0}^{t_1}H(t_2)dt_2 \\ &\quad + \left(\frac{-i}{\hbar}\right)^3\int_{t_0}^{t}H(t_1)dt_1\int_{t_0}^{t_1}H(t_2)dt_2\int_{t_0}^{t_2}H(t_3)dt_3 + \cdots\,. \end{aligned} \tag{24}$$

So, if we take the time derivative, we'll get (3). However, this equation is of limited usefulness.

Now that we know $U(t,t_0)$, we can evolve the wave function in time.

2 The Heisenberg Picture of Quantum Mechanics

We begin with the Scrödinger picture of quantum mechanics. It contains operators such as x, p, spin, and the Hamiltonian, wave functions, onto which we develop a new way to think about it.

Consider a generic Scrödinger operator $\widehat{A}_S$. What is the matrix element between these two states $|\,\alpha,t\,\rangle$ and $|\,\beta,t\,\rangle$?

$$\Big\langle\,\alpha,t\,|\,\widehat{A}_S\,|\,\beta,t\,\Big\rangle = \Big\langle\,\alpha,0\,|\,U^{\dagger}(t,0)\widehat{A}_S U(t,0)\,|\,\beta,0\,\Big\rangle\,. \tag{25}$$

Now, we have a dual way to interpret this. On the LHS, we have the effect of $\widehat{A}_S$ and on the RHS, we have the effect of the time-dependent operator $U^{\dagger}(t,0)\widehat{A}_S U(t,0)$ on the time-independent states $|\,\alpha,0\,\rangle$ and $|\,\beta,0\,\rangle$.

Let us define the new operator

$$\widehat{A}_H \equiv U^{\dagger}(t,0)\widehat{A}_S U(t,0)\,, \tag{26}$$

Comments:

At $t = 0$, $\widehat{A}_H(t=0) = \widehat{A}_S\ (t=0)$.

$\mathbb{1}_S \longrightarrow \mathbb{1}_H = U^\dagger(t,0)\mathbb{1}_S U(t,0) = \mathbb{1}_S$.

Problem: Given the operator $\widehat{C}_S$ such that

$$\widehat{C}_S = \widehat{A}_S \widehat{B}_S\,, \tag{27}$$

what is $\widehat{C}_H$?

$$\widehat{C}_H = U^\dagger \widehat{C}_S U = U^\dagger \widehat{A}_S U U^\dagger \widehat{B}_S U = \widehat{A}_H \widehat{B}_H\,, \tag{28}$$

If $[\,\widehat{A}_S, \widehat{B}_S\,] = \widehat{C}_S$ then $[\,\widehat{A}_H, \widehat{B}_H\,] = \widehat{C}_H$.

Since $[\,x, p\,] = i\hbar\mathbb{1}$ then $[\,x_H(t), p_H(t)\,] = i\hbar\mathbb{1}$.

Now, what about Hamiltonians?

$$H_H = U^\dagger(t,0) H_S U(t,0)\,. \tag{29}$$

And for all t_1, t_2,

$$[\,H_S(t_1), H_S(t_2)\,] = 0\,, \tag{30}$$

then

$$H_H(t) = H_S(t)\,. \tag{31}$$

But if $H_S(t) = H_H(\widehat{x}, \widehat{p}, t)$ then

$$H_H(t) = U^\dagger H_S(t) U = U^\dagger H_S(\widehat{x}_S, \widehat{p}_S, t) U = H_S(\widehat{x}_H, \widehat{p}_H, t) = H_S(\widehat{x}, \widehat{p}, t)\,, \tag{32}$$

established by the usual means. This is a useful result.

Expectation Values

Set both α and β equal to $|\,\psi, t\,\rangle$:

$$\Big\langle\, \psi, t\ |\ \widehat{A}_S\,|\,\psi, t\,\Big\rangle = \Big\langle\, \psi, 0\ |\ \widehat{A}_H(t)\,|\,\psi, 0\,\Big\rangle\,. \tag{33}$$

This can simplify some computations. In shorthand:

$$\langle\, \widehat{A}_S\,\rangle = \langle\, \widehat{A}_H(t)\,\rangle\,, \tag{34}$$

is used but needs some interpretation.

And we're back to the problem of determining $\widehat{A}_H$ where U is difficult to calculate.

Heuristic: Try to find a differential equation that is satisfied by the Heisenberg operator, other than

$$\widehat{A}_H = U^\dagger(t,0)\widehat{A}_S U(t,0)\,, \tag{35}$$

So,

$$i\hbar\frac{d}{dt}\widehat{A}_H = i\hbar\frac{\partial U^\dagger}{\partial t}\widehat{A}_S U + i\hbar U^\dagger\frac{\partial \widehat{A}_S}{\partial t}U + i\hbar U^\dagger \widehat{A}_S\frac{\partial U}{\partial t}\,. \tag{36}$$

But

$$i\hbar\frac{\partial U}{\partial t} = H_S U\,, \tag{37}$$

So,

$$i\hbar\frac{\partial U^\dagger}{\partial t} = U^\dagger H_S\,, \tag{38}$$

where $i^\dagger = -i$. Hence,

$$i\hbar\frac{d}{dt}\widehat{A}_H = -U^\dagger H_S \widehat{A}_S U + U^\dagger \widehat{A}_S H_S U + i\hbar\left(\frac{\partial \widehat{A}_H}{\partial t}\right)_H\,, \tag{39}$$

This can be rewritten as

$$i\hbar\frac{d\widehat{H}_H(t)}{dt} = [\,\widehat{A}_H, \widehat{H}_H\,] + i\hbar\left(\frac{\partial \widehat{A}_S}{\partial t}\right)_H\,, \tag{40}$$

This is the Heisenberg equation of motion. Let's go to cases.

1) Suppose $\dfrac{\partial \widehat{A}_S}{\partial t} \equiv 0$ then

$$i\hbar\frac{d\widehat{H}_H(t)}{dt} = [\,\widehat{A}_H, \widehat{H}_H(t)\,]\,. \tag{41}$$

2) $\widehat{A}_S$ has no explicit time dependence.

$$\begin{aligned} i\hbar\frac{d}{dt}\left\langle\,\Psi, t\mid\widehat{A}_S\mid\Psi, t\,\right\rangle &= i\hbar\frac{d}{dt}\left\langle\,\Psi, 0\mid\widehat{A}_H\mid\Psi, 0\,\right\rangle \\ &= \left\langle\,\Psi, 0\mid i\hbar\frac{d\widehat{A}_H}{dt}\mid\Psi, 0\,\right\rangle \\ &= \left\langle\,\Psi, 0\mid[\,\widehat{A}_H, \widehat{H}_H\,]\mid\Psi, 0\,\right\rangle \end{aligned} \tag{42}$$

From this we get that

$$i\hbar\frac{d}{dt}\langle\,\widehat{A}_H(t)\,\rangle = \langle\,[\,\widehat{A}_H, \widehat{H}_H\,]\,\rangle\,, \tag{43}$$

or

$$i\hbar\frac{d}{dt}\langle\,\widehat{A}_S\,\rangle = \langle\,[\,\widehat{A}_S, \widehat{H}_S\,]\,\rangle\,. \tag{44}$$

Is it a conserved operator? $\widehat{A}_S$ is conserved if it commutes with the Hamiltonian. $[\,\widehat{A}_S, \widehat{H}_S\,] = 0$. But this implies that

$$[\,\widehat{A}_H, \widehat{H}_H\,] = 0\,, \tag{45}$$

which then implies that

$$\frac{d\langle\,\widehat{A}_H\,\rangle}{dt} = 0 \quad \text{which implies that} \quad \frac{d\widehat{A}_H}{dt} = 0\,. \tag{46}$$

In this case we get that $\widehat{A}_H$ is time-independent.

Example: Harmonic oscillator.

$$H_S = \frac{\widehat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\widehat{x}^2\,. \tag{47}$$

Or,

$$H_H = \frac{\widehat{P}_H^2}{2m} + \tfrac{1}{2} m\omega^2 \widehat{X}_H^2(t)\,. \tag{48}$$

Can we solve for $\widehat{X}_H$ and $\widehat{P}_H$?

$$i\hbar \frac{d\widehat{X}_H}{dt} = [\,\widehat{X}_H, \widehat{H}_H\,] = [\,\widehat{X}_H, \frac{1}{2m}\widehat{P}_H^2\,] = \frac{1}{2m}\widehat{P}_H(i\hbar)2\,. \tag{49}$$

Yielding,

$$\frac{d\widehat{X}_H}{dt} = \frac{1}{m}\widehat{P}_H\,, \tag{50}$$

which looks like classical mechanics. Also,

$$i\hbar \frac{d\widehat{P}_H}{dt} = [\,\widehat{P}_H, \widehat{H}_H\,] = [\,\widehat{P}_H, \tfrac{1}{2}m\omega^2\widehat{X}_H^2\,] \; = \tfrac{1}{2}m\omega^2 2\widehat{X}_H(-i\hbar)\,. \tag{51}$$

Hence,

$$\frac{d\widehat{P}_H}{dt} = -m\omega^2\widehat{X}_H\,. \tag{52}$$

On differentiating, we get

$$\frac{d^2\widehat{X}_H}{dt^2} = \frac{1}{m}\frac{d\widehat{P}_H}{dt} = \frac{1}{2m}(-m\omega^2\widehat{X}_H)\,. \tag{53}$$

Finally,

$$\frac{d^2\widehat{X}_H}{dt^2} = -\omega^2\widehat{X}_H\,, \tag{54}$$

with solutions

$$\widehat{X}_H(t) = \widehat{A}\cos\omega t + \widehat{B}\sin\omega t\,, \tag{55}$$

where $\widehat{A}$ and $\widehat{B}$ are time-independent operators. Similarly,

$$\widehat{P}_H(t) = m\frac{d\widehat{X}}{dt} = -m\omega\widehat{A}\sin\omega t + m\omega\widehat{B}\cos\omega t\,. \tag{56}$$

At $t = 0$,

$$\widehat{X}_H(t=0) = \widehat{A} = \widehat{X}\,, \tag{57}$$

and

$$\widehat{P}_H(t=0) = m\omega\widehat{B} = \widehat{P}\,, \tag{58}$$

implying that

$$\widehat{B} = \frac{1}{m\omega}\widehat{P}\,. \tag{59}$$

This leaves us with the complete solution.

$$\widehat{X}_H(t) = \widehat{X}\cos\omega t + \frac{\widehat{P}}{m\omega}\sin\omega t\,, \tag{60a}$$

$$\widehat{P}_H(t) = \widehat{P}\cos\omega t - m\omega\widehat{X}\sin\omega t\,. \tag{60b}$$

Finally, let's calulate the Heisenberg Hamiltonian.

$$\begin{aligned}
\widehat{H}_H(t) &= \frac{1}{2m}(\widehat{P}\cos\omega t - m\omega\widehat{X}\sin\omega t)^2 + \tfrac{1}{2}m\omega^2(\widehat{X}\cos\omega t + \frac{\widehat{P}}{m\omega}\sin\omega t)^2 \\
&= \frac{1}{2m}\cos^2\omega t\,\widehat{P}^2 + \frac{1}{2m}m^2\omega^2\sin^2\omega t\,\widehat{X}^2 - \frac{1}{2m}m\omega\sin\omega t\cos\omega t\,(\widehat{P}\widehat{X} + \widehat{X}\widehat{P}) \\
&= \frac{\frac{1}{2}m\omega^2}{m^2\omega^2}\sin^2\omega t\widehat{P}^2 + \tfrac{1}{2}m\omega^2\cos^2\omega t\widehat{X}^2 + \tfrac{1}{2}\frac{m\omega^2}{m\omega}\cos\omega t\sin\omega t\,(\widehat{X}\widehat{P} + \widehat{P}\widehat{X}) \\
&= \frac{\widehat{P}^2}{2m} + \tfrac{1}{2}m\omega^2\cos^2\widehat{X}^2 = H_S(t)\,.
\end{aligned} \tag{61}$$