

Quantum Mechanics Notes for A. Adams's Lecture Series.

Lecture 13: More on Scattering

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Abstract

This paper contains my read-along notes on Lecture Thirteen of Allan Adams's 2013 presentation on Quantum Mechanics for his MIT Video Lecture Series (8.04). These notes are meant to aid the reader in following Prof. Adams's presentation, without having to take copious notes. The fault for any inaccuracies in this paper belongs to the author.

1 Some review

We begin with the case $E < V_0$. In the figure below we compare to the classical prediction.

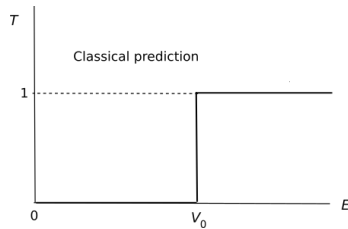


Figure 1. The classical prediction of transmission. The particle enters from the left. In the classical prediction, no transmission will occur when $E < V_0$.

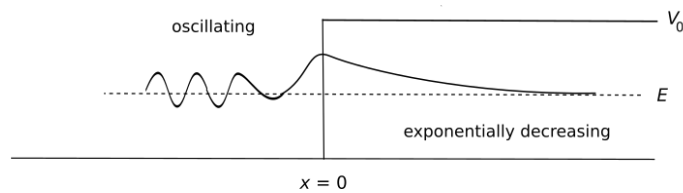


Figure 2. A free particle of energy E encounters a step potential V_0 . ($E - V_0 < 0$).

$$\phi_E = A \begin{cases} e^{ikx} + \frac{k - i\alpha}{k + i\alpha} e^{-ikx} & \text{left side,} & \frac{\hbar^2 k^2}{2m} = E, \\ \frac{2k}{k + i\alpha} e^{-\alpha x} & \text{right side,} & \frac{\hbar^2 \alpha^2}{2m} = V_0 - E. \end{cases} \quad (1)$$

We construct our wavepacket on the right ψ_R as:

$$\psi_R = \frac{2k}{k+i\alpha} e^{-\alpha x}, \quad \psi_R^* = \frac{2k}{k-i\alpha} e^{-\alpha x}. \quad (2)$$

And

$$\partial_x \psi_R = \frac{-2k\alpha}{k+i\alpha} e^{-\alpha x}, \quad \partial_x \psi_R^* = \frac{-2k}{k-i\alpha} e^{-\alpha x}. \quad (3)$$

Therefore,

$$J_T = \frac{\hbar}{2mi} (\psi^* \partial_x \psi - \psi \partial_x \psi^*) \sim \frac{2k}{k-i\alpha} e^{-\alpha x} \left[\frac{-2k}{k+i\alpha} e^{-\alpha x} \right] - \frac{2k}{k+i\alpha} e^{-\alpha x} \left[\frac{-2k}{k-i\alpha} e^{-\alpha x} \right] = 0. \quad (4)$$

$$T = \left| \frac{J_T}{J_I} \right|^2, \quad R = \left| \frac{J_R}{J_I} \right|^2. \quad (5)$$

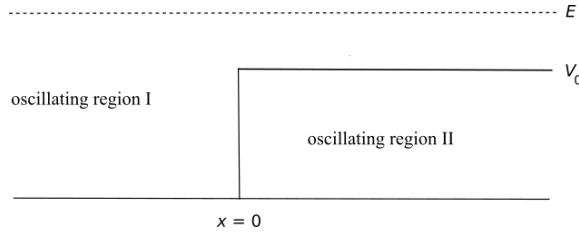


Figure 3. A free particle encounters a step potential ($E > V_0$). We expect oscillating solutions on both sides of the jump up.

Note: Left/Right and In/Out have their meanings relative to the barrier step-up.

$$\phi_E = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & \text{left side,} & \frac{\hbar^2 k_1^2}{2m} = E, \\ Ce^{ik_2x} + De^{-ik_2x} & \text{right side,} & \frac{\hbar^2 k_2^2}{2m} = E - V_0, \end{cases} \quad (6)$$

How then do these energy eigenstates evolve in time?

$$\phi_E = \begin{cases} Ae^{i(k_1x - \omega t)} + Be^{-i(k_1x + \omega t)} & \text{left side,} & \frac{\hbar^2 k_1^2}{2m} = E, \\ Ce^{i(k_2x - \omega t)} + D e^{-i(k_2x + \omega t)} & \text{right side,} & \frac{\hbar^2 k_2^2}{2m} = E - V_0, \end{cases} \quad (7)$$

where $e^{i(kx - \omega t)}$ is right-moving and $e^{-i(kx + \omega t)}$ is left-moving. We have set D to zero because there will be no waves moving from right to left in Region II. If, instead, we set up the experiment so that the particle enters from the right, then $A = 0$ and $D \neq 0$.

With some algebra, we get:

$$C = \frac{2k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}. \quad (8)$$

And,

$$R = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2, \quad T = \frac{4k_1 k_2}{|k_1 + k_2|^2}. \quad (9)$$

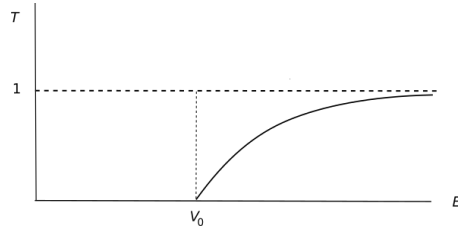


Figure 4. The quantum mechanical picture of transmission at a step function potential. Note that as E goes much larger than V_0 , transmission is near certain.

Let's rewrite (9) into the form

$$R = \left| \frac{1 - \sqrt{1 - V_0/E}}{1 + \sqrt{1 - V_0/E}} \right|^2, \quad T = \frac{4\sqrt{1 - V_0/E}}{\left| 1 + \sqrt{1 - V_0/E} \right|^2}, \quad (10)$$

where $R + T = 1$. To an experimentalist, the shape of the T versus E graph can tell a lot about the nature of the potential being probed.

Alternatively, for a particle coming in from the right, we have that $A = 0$, and that

$$C = \frac{k_1 - k_2}{k_1 + k_2} D, \quad B = \frac{2k_1}{k_1 + k_2} D, \quad (11)$$

and R, T are the same.

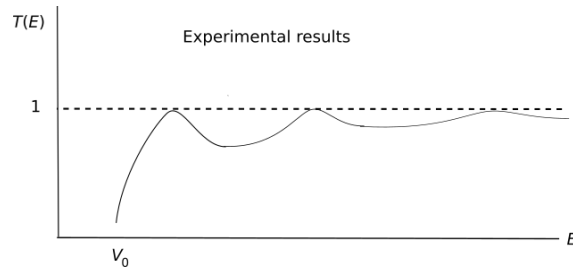


Figure 5. A possible experimental-result graph.

2 Probing the unknown potential

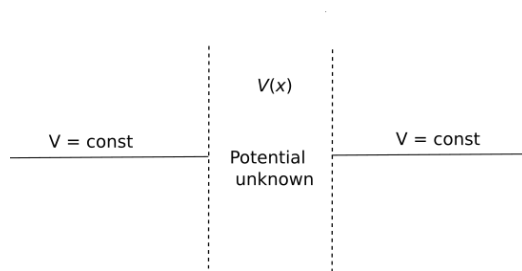


Figure 6. There's always something to say about an unknown potential in this configuration. Can we infer $V(x)$ by probing the potential with particles of well-defined energy E ?

Referring to Fig. 6, on the left we have $Ae^{ikx} + Be^{-ikx}$ and on the right, $Ce^{ikx} + De^{-ikx}$. The transmission and reflection values are given (asymptotically) as

$$T = \left| \frac{C}{A} \right|^2, \quad R = \left| \frac{B}{A} \right|^2. \quad (12)$$

To calculate T, R , it suffices to know B, C in terms of A, D .

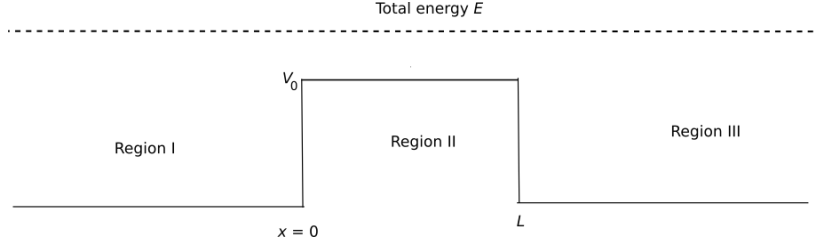


Figure 7. The particle encounters a finite-width rectangular barrier, having total energy greater than the potential step it encounters. Depicted is the well-defined regions of oscillatory behavior.

Since we have scattering from an incoming particle from the left, we set $D = 0$.

Case I: $E > V_0$.

Region	Ψ
I	$Ae^{ikx} + Be^{-ikx}$
II	$F e^{ik'x} + G e^{-ik'x}$
III	$C e^{ikx} + \cancel{D} e^{-ikx}$

And

$$\frac{\hbar^2 k^2}{2m} = E \text{ (Regions I,III)}, \quad \frac{\hbar^2 k'^2}{2m} = E - V_0 \text{ (Region II)}. \quad (13)$$

On meeting the boundary conditions, we get with much algebra:

$$T = \left| \frac{C}{A} \right|^2 = \frac{4k^2 k'^2}{4k^2 k'^2 \cos^2(k'L) + (k^2 + k'^2) \sin^2(k'L)}. \quad (14)$$

We can simplify this expression by introducing the parameters:

$$g_0^2 \equiv \frac{2mL^2 V_0}{\hbar^2}, \quad (15)$$

$$\epsilon \equiv \frac{E}{V_0}. \quad (16)$$

With these, (14) becomes

$$T = \frac{1}{1 + \frac{1}{4\epsilon(\epsilon - 1)} \sin^2(g_0 \sqrt{\epsilon - 1})}. \quad (17)$$

Hence, for the case $\epsilon \rightarrow 1$

$$T \rightarrow \frac{1}{1 + g_0^2/4}. \quad (18)$$

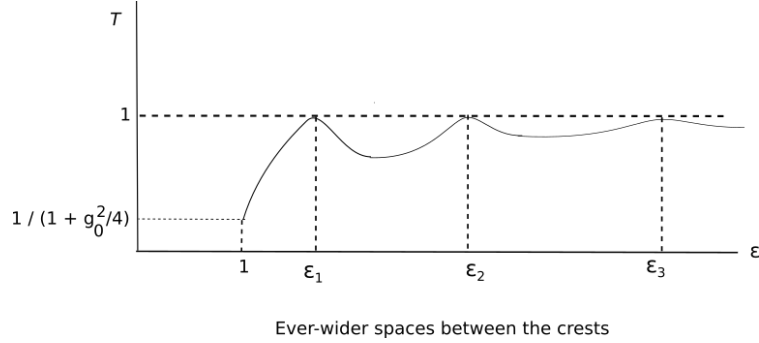


Figure 8. The points where maximal transmissions occur are called *resonances*.

If an experimentalist experiences a transmission curve as that in the above figure, it may indicate a rectangular potential.

Case I: $E < V_0$.

In this case only the only change occurs in the middle section, Region II, where

$$\psi_{II} = F e^{-\alpha x} + G e^{\alpha x}. \quad (19)$$

This gives us the value $\epsilon < 1$.

$$T = \frac{1}{1 + \frac{1}{4\epsilon(1-\epsilon)} \sinh^2(g_0 \sqrt{1-\epsilon})}. \quad (20)$$

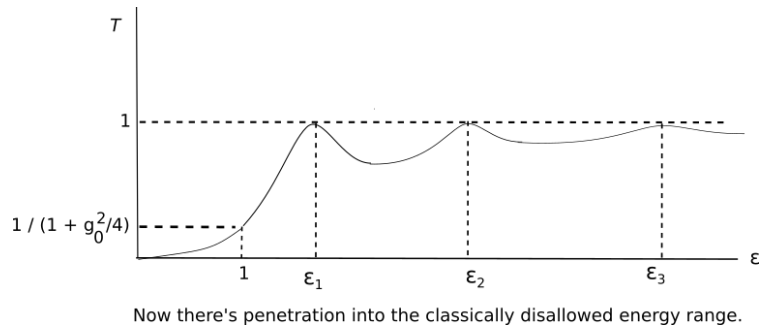


Figure 9. The classically disallowed energy regime is now possible.

So, in this case we have transmission into the classically disallowed energy range.

Holding E fixed, what happens when we vary L ? For $L \gg 1$, $T \sim e^{-2\alpha L}$, which gets less and less likely as the width L increases.