Quantum Mechanics Notes for A. Adams's Lecture Series.
Lecture 23: More on Spin states

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Abstract

This paper contains my read-along notes on Lecture 23 of Allan Adams's 2013 presentation on Quantum Mechanics for his MIT Video Lecture Series (8.04). These notes are meant to aid the reader in following Prof. Adams's presentation, without having to take copious notes. The fault for any inaccuracies in this paper belongs to the author.

1 Review of orbital angular momenum and then to its spin analogy

However we imagine the nature of particle spin, we have to grant it some angular momentum. It is therefore logical to formulate spin angular momentum in parallel to orbital angular momentum, which we'll review now.

We have for the relations for orbital angular momentum,

$$\widehat{L}_z = \widehat{x}p_y - \widehat{y}p_x \,, \tag{1}$$

and we note that $\widehat{L}_z$ is hermitian. If we make the identifications:

$$\widehat{L}_x = \widehat{L}_1 \,, \quad \widehat{L}_y = \widehat{L}_2 \,, \quad \widehat{L}_z = \widehat{L}_3 \,, \tag{2}$$

then we get this simplifying relations

$$[\,\widehat{L}_i, \widehat{L}_j\,] = i\hbar\epsilon_{ijk}\widehat{L}_k \,. \tag{3}$$

From this we get that:

$$\begin{aligned} [\,\widehat{L}_x, \widehat{L}_y\,] &= i\hbar\widehat{L}_z \,, \\ [\,\widehat{L}_y, \widehat{L}_z\,] &= i\hbar\widehat{L}_x \,, \\ [\,\widehat{L}_z, \widehat{L}_x\,] &= i\hbar\widehat{L}_y \,. \end{aligned} \tag{4}$$

In analogy to these relations, we propose for spin angular momentum the relations:

$$\begin{aligned} [\,\widehat{S}_x, \widehat{S}_y\,] &= i\hbar\widehat{S}_z \,, \\ [\,\widehat{S}_y, \widehat{S}_z\,] &= i\hbar\widehat{S}_x \,, \\ [\,\widehat{S}_z, \widehat{S}_x\,] &= i\hbar\widehat{S}_y \,. \end{aligned} \tag{5}$$

By analogy with (3), we expect that,

$$[\,\widehat{S}_i,\,\widehat{S}_j\,] = i\hbar\epsilon_{ijk}\widehat{S}_k\,. \tag{6}$$

These spin operators can have half-integer values.

Furthermore, we have the familair relations:

$$\widehat{S}_\pm = \widehat{S}_x \pm i\hbar\widehat{S}_y\,, \tag{7a}$$

$$\widehat{S}^2 = \hbar s(s+1) \quad \text{where} \quad s \sim \frac{2n+1}{2}\,, \tag{7b}$$

$$\widehat{S}_z = \hbar m_s \quad \text{where} \quad -s \leq m_s \leq s\,. \tag{7c}$$

For spin-$\frac{1}{2}$ states, we adopt the convention to represent the two possible anti-aligned paired states in a magnetic field as ↑, ↓. For a superposition of spin states:

$$\psi(x) = \psi_{+\frac{1}{2}}(x) + \psi_{-\frac{1}{2}}(x)\,. \tag{8}$$

The probability at point x of obtaining a measurement of either $\pm\hbar/2$ is:

$$P(x, \pm\tfrac{1}{2}\hbar) = |\,\psi_\pm(x)\,|^2\,. \tag{9}$$

2 Spin represented with matrices

Notation: The dagger will mean hermitian conjugation.

Given

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} \quad \text{then}\,, \tag{10a}$$

$$\Psi^\dagger(x) = \left(\psi^*_\uparrow(x)\,,\ \psi^*_\downarrow(x)\right)\,, \tag{10b}$$

where Ψ is referred to as a two-component *spinor*. For the norm, we have

$$\langle\,\Psi \mid \Psi\,\rangle = 1 = \int dx \left(\,|\,\psi_\uparrow(x)\,|^2 + |\,\psi_\downarrow(x)\,|^2\,\right). \tag{11}$$

If we begin with the superposition

$$\Psi(x) = \psi_\uparrow(x)\,|\!\uparrow\rangle + \psi_\downarrow(x)\,|\!\downarrow\rangle\,, \tag{12}$$

we can now represent $|\!\uparrow\rangle$ and $|\!\downarrow\rangle$ with 2×1 matrices as follows

$$\Psi(x) = \psi_\uparrow(x)\begin{pmatrix}1\\0\end{pmatrix} + \psi_\downarrow(x)\begin{pmatrix}0\\1\end{pmatrix}\,. \tag{13}$$

To represent Eq. (12) in Dirac notation, we get

$$|\,\Psi(x)\,\rangle = |\,\psi_\uparrow(x)\,\rangle\,|\!\uparrow\rangle + |\,\psi_\downarrow(x)\,\rangle\,|\!\downarrow\rangle\,, \tag{14}$$

3 Some relations we need to know

Okay, those are the states, what about the representation of the operators? Let's for similarities and differences to the operators on nonspin wavefunctions:

$$L_z = \frac{\hbar}{i}\partial_\phi \,, \tag{15a}$$

$$[\, L_x, L_y \,] = i\hbar L_z \,. \tag{15b}$$

However, the spin operators cannot be represented by derivatives.[1]

The analogy for spin operators for spin equations are

$$S_z \,|\!\uparrow\rangle = \frac{\hbar}{2}\,|\!\uparrow\rangle \ , \quad S_z \,|\!\downarrow\rangle = -\frac{\hbar}{2}\,|\!\downarrow\rangle \ , \tag{16}$$

and

$$S^2 \,|\!\uparrow\rangle = \frac{3\hbar^2}{4}\,|\!\uparrow\rangle \quad \text{and} \quad S^2\,|\!\downarrow\rangle = \frac{3\hbar^2}{4}\,|\!\downarrow\rangle \ . \tag{17}$$

Another correspondence goes as:

$$L^2\psi_{\ell m} = \hbar^2\ell(\ell+1)\psi_{\ell m} \,, \tag{18}$$

$$S^2\psi_{sm_s} = \hbar^2 s(s+1)\psi_{sm_s} \quad \text{with} \quad s = 1/2 \,. \tag{19}$$

Also

$$S_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix} , \qquad S_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} , \tag{20}$$

$$S^2\begin{pmatrix}0\\1\end{pmatrix} = \frac{3\hbar}{4}\begin{pmatrix}0\\1\end{pmatrix} , \qquad S^2\begin{pmatrix}0\\1\end{pmatrix} = \frac{3\hbar}{4}\begin{pmatrix}0\\1\end{pmatrix} . \tag{21}$$

What we soon find out is that the action of the other operators S_x and S_y on their states can be expressed in terms of states ↑ and ↓. Anyway, we begin with the operator that acts on ↑ and ↓, which are its eigenvectors.

$$\widehat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} . \tag{22}$$

and with this:

$$\widehat{S}^2\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} . \tag{23}$$

$$\widehat{S}^2 = s(s+1)\,. \tag{24}$$

Also, in matrix formalism:

$$S_+\begin{pmatrix}0\\1\end{pmatrix} = \hbar\begin{pmatrix}1\\0\end{pmatrix} , \qquad S_-\begin{pmatrix}0\\1\end{pmatrix} = 0 \tag{25}$$

$$S_+\begin{pmatrix}1\\0\end{pmatrix} = 0\,, \qquad S_-\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\1\end{pmatrix} . \tag{26}$$

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[1] Why is the z direction used as the starting direction for the analysis of spin and does this direction have to define up and down in the laboratory? In answer to the second question, I'd say no, but by convention it seems to. This convention predates quantum mechanics and goes back to classical physics. In the lab, up and down is (typically) defined by, or conforms to, the z axis.

Given the above relations, what should $S_\pm$ look like as a matrix?

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{27}$$

Given that,

$$S_x = \tfrac{1}{2}(S_+ + S_-) , \quad S_y = \frac{1}{2i}(S_+ - S_-) , \tag{28}$$

and as matrices

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S_y = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{29}$$

Exercise: Using matrices, show that $[\, S_x, S_y \,] = i\hbar S_z$.

Now, let's do for S_x and S_y what we did for S_z:

$$S_x \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad S_x \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{30}$$

In arrow notation:

$$\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z) . \tag{31}$$

This holds valuable information, for if the spin is in the x direction, another measurement taken instead in the z direction, gives a 50-50 chance of being measured in the z up or z down directions. Likewise,

$$\downarrow_x = \frac{1}{\sqrt{2}}(\uparrow_z - \downarrow_z) . \tag{32}$$

Hence,

$$P(\uparrow_z, \downarrow_x) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} . \tag{33}$$

Doing the same thing for S_y, we have that

$$\uparrow_y = \frac{1}{\sqrt{2}}(\uparrow_z + i \downarrow_z) , \quad \downarrow_y = \frac{1}{\sqrt{2}}(\uparrow_z - i \downarrow_z) . \tag{34}$$

For later use, we introduce the operator S_θ.

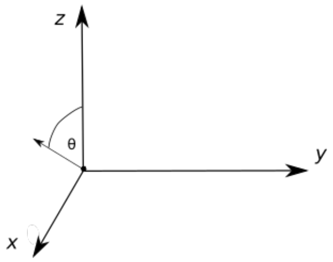

Figure 1. Theta is an angle in the xz-plane. What is the s spin operator along the direction given by θ?

$$\uparrow_\theta = \cos\left(\tfrac{1}{2}\theta\right)\uparrow_z + \sin\left(\tfrac{1}{2}\theta\right)\downarrow_z\,, \tag{35}$$

and

$$\downarrow_\theta = \sin\left(\tfrac{1}{2}\theta\right)\uparrow_z - \cos\left(\tfrac{1}{2}\theta\right)\downarrow_z\,. \tag{36}$$

Show that $\uparrow_\theta$ and $\downarrow_\theta$ are orthogonal.

4 Daisy-chained Stern-Gerlach Apparatus

Suppose we measure S_z to get the value $\hbar/2$. The state of the system ψ after this measurement is $\psi = \uparrow_z$. But

$$\uparrow_z = \frac{1}{\sqrt{2}}(\uparrow_x + \downarrow_x)\,. \tag{37}$$

If we then 'measure S_x', then, since,

$$P(S_x = +\frac{\hbar}{2}, \uparrow_z) = \tfrac{1}{2}\,, \quad P(S_x = -\frac{\hbar}{2}, \uparrow_z) = \tfrac{1}{2}\,. \tag{38}$$

A similar result occurs for measuring S_y.

Now, suppose we set up an experimental apparatus as in the following figure.

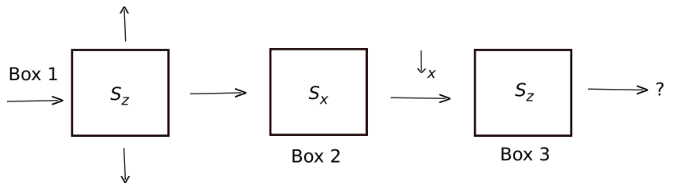

Figure 2. An electron enters from the left into Box 1 which has a magnetic field in the z direction. After leaving Box 1, the particle enters Box 2, which has a magnetic field on in the x direction. On average, the exiting electron will have equal chance to exit in the $+x$ direction as in the $-x$ direction. If we then let the down x electrons pass through another S_z apparatus, we expect 50-50 output of up z and down z.

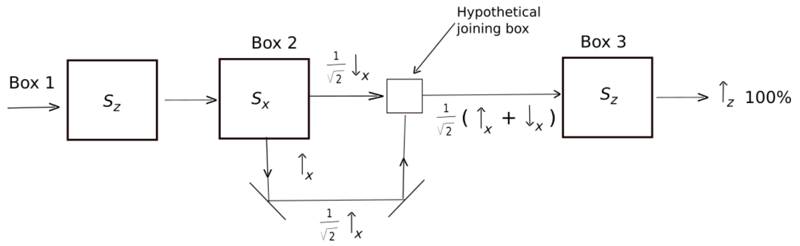

Figure 3. This time we place a 'joining box' between Boxes 2 and 3, which has the effect of sending the supperposition $\frac{1}{\sqrt{2}}(\uparrow_x + \downarrow_x)$ into Box 3. The result coming out of Box 3 is 100% $\uparrow_z$.

5 Mathematics of the Stern-Gerlach experiment

The Stern-Gerlach apparatus has a clever way to particle spin with physical movenment via a special non-uniform external magnetic field, but the field is zero outside the apparatus. Let's see how this is done. First, we set the special magnetic field B_z according to the rule

$$B_z = B_0 + \beta z \,, \tag{39}$$

where B_0 and β are constants, and where βz represents a small gradient in the otherwise uniform magnetic field. From this we get the wavefunction

$$\psi_e = a \uparrow_z + b \downarrow_z \ . \tag{40}$$

The energy of this interaction is given as

$$E = -\mu_0 \mathbf{s} \cdot \mathbf{B} = -\mu_0 s_z (B_0 + \beta z) \,, \tag{41}$$

which is the energy in the region of the Stern-Gerlach apparatus.

Now, we focus on phases.

$$E = -\mu_0 s_z (B_0 + \beta z) = -\mu_0 (B_0 + \beta z) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c(z) & 0 \\ 0 & -c(z) \end{pmatrix} \,, \tag{42}$$

where we introduced the function $c(z)$ for convenience. Anyway,

$$E \uparrow_z = c(z) \uparrow_z \,, \tag{43a}$$

$$E \downarrow_z = -c(z) \downarrow_z \ . \tag{43b}$$

So, we have found that $\uparrow_z$ and $b \downarrow_z$ are eigenvector of the energy operator, with

$$E_{\uparrow_z} = +c(z) \,, \quad E_{\downarrow_z} = -c(z) \,. \tag{44}$$

The origin of this energy-splitting degeneracy is that in the zero-magnetic field range, the electron wavefunction has full spherical symmetry, but inside the Stern-Gerlach region, is broken by introducing a preferred direction (the z direction).

But

$$\Psi(t) = a e^{iE_\uparrow t/\hbar} \uparrow_z + b e^{-iE_\downarrow t/\hbar} \downarrow_z \tag{45}$$

$$= a\Big(e^{i\mu_0 B_0 t/2} e^{\mu_0 \beta t z/2}\Big) \uparrow_z + b\Big(e^{-i\mu_0 \beta t/2} e^{-\mu_0 \beta t z/2}\Big) \downarrow_z \ . \tag{46}$$

Note that the state $\phi_{k_z} = e^{ik_z z}$ carries $\hbar k_z$ amount of momentum. So, we see that, as the electron traverses the Stern-Gerlach apparatus, it will pick up mometum either in the $+z$ direction or in the $+z$ direction, consistent with the state vector orientation along the z axis.