Quantum Mechanics Notes for A. Adams's Lecture Series. Lecture 24: Entanglement: QComputing, EPR, and Bell's Theorem

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Abstract

This paper contains my read-along notes on Lecture 24 of Allan Adams's 2013 presentation on Quantum Mechanics for his MIT Video Lecture Series (8.04). These notes are meant to aid the reader in following Prof. Adams's presentation, without having to take copious notes. The fault for any inaccuracies in this paper belongs to the author.

1 Review

We have for the relations for spin angular momentum,

$$[\widehat{S}_x, \widehat{S}_y] = i\hbar \widehat{S}_z \,. \tag{1}$$

$$\psi(0) = \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z)\,. \tag{2}$$

$$E = -\mu B_z S_z \Longrightarrow E_\pm = \pm\hbar\omega,\ \omega = \frac{\mu B_z}{2}\,. \tag{3}$$

$\Longrightarrow$

$$\psi(t) = \frac{1}{\sqrt{2}}(e^{i\omega t}\uparrow_z + e^{-i\omega t}\downarrow_z)\,. \tag{4}$$

If we turn on the magnetic field in the z direction for a time T, given by

$$\omega T = \frac{\pi}{2}\,, \tag{5}$$

we end up with

$$\Psi_{\text{after}} = \frac{i}{\sqrt{2}}(\uparrow_z - \downarrow_z) = i\downarrow_x \,. \tag{6}$$

The spin has precessed around the z axis. Hence, to convert one spin state into another, apply an appropriate magnetic field.

Any unitary operator that takes one spinor to another can be constructed in an apparatus by a collection of magnetic fields.

Hence,

1. System $\uparrow_z$, $\downarrow_z$.
2. Ability to evolve states from $\psi_1 \to \psi_2$ by some applied magnetic field.
3. What can we do with this ability?

Answer: For one, we can build a quantum computer.

2 Quantum Computing

1) Classical bits (0 or 1) states.
Use classical mechanics to buiild a machine.

Example: Calculation:

$$1\ 1\ 0\ 0 \xrightarrow{f(\text{in})} 0\ 0\ 1\ 1\ .$$

For N bits we can represent a string of N bits.

Now, we compare to quantum bits or Qbits: $\{|\,0\,\rangle\ ,\ |\,1\,\rangle\}$ to represent a particle state, we have that

$$\psi = \alpha\,|\,0\,\rangle\ + \beta\,|\,1\,\rangle \quad \text{(a superposition)}\,. \tag{7}$$

It takes two complex number to represent the state of a Qbit. For N Qbits: $\{|\,0\,\rangle\ ,\ |\,1\,\rangle\}^N$:

$$\psi = \alpha\,|\,0000\ldots0\,\rangle\ + \beta\,|\,1000\ldots0\,\rangle\ + \ldots\,. \tag{8}$$

In this case we need 2^N complex numbers. To store in memory the information for 10 Qbits, we'd need 2^{10} bits.

However, it's easy for the quantum system to emulate a classical system because the real world is classical but the underlying reality is quantum.

Can we use quantum evolution to perform computation efficiently?

System: N bits $\overset{1}{\uparrow}$, $\overset{0}{\downarrow}$, so that

In: $\psi_{\text{in}}^{(N)}$ which we let evolve with $\widehat{E}$ to implement some desired algorithm. Then:

Out: $\psi_{\text{out}}^{(N)}$

We take note that this system input is in a superposition of states.

2) During measurement, the various terms mutually interfere with each other.

3) Therefore, expect probability outcomes.

4) What we need to do is to figure out some clever way to direct these interfences to get the desired outcomes.

5) Focus our attention on checkable problems.

The key to a successful evolution will require the entanglement of states.

3 Getting down to it

A Qbit is a two-state system: $|\,0\,\rangle$ or $|\,1\,\rangle$.

$$\psi = \alpha\,|\,0\,\rangle\ + \beta\,|\,1\,\rangle\ , \tag{9}$$

where

$$P(0) = |\,\alpha\,|^2\ , \quad P(1) = |\,\beta\,|^2\ . \tag{10}$$

Now, for 2 Qbits, the first one represented by A, the second one represented by B:

$$\phi = \alpha\,|\,00\,\rangle + \beta\,|\,01\,\rangle + \gamma\,|\,10\,\rangle + \delta\,|\,11\,\rangle \ , \tag{11}$$

where

$$P(A=0) = |\,\alpha\,|^2 + |\,\beta\,|^2 \ , \quad \text{and} \quad P(A=0, B=1) = |\,\beta\,|^2 \ . \tag{12}$$

There two kinds of states that pair of Qbits can be in: separable and nonseparable. The separable states correspond to

$$|\,\psi\,\rangle = |\,\psi_1\,\rangle\,|\,\psi_2\,\rangle \ , \tag{13}$$

where particle 1 is in state 1 and particle 2 is in state 2.

Example:

$$\begin{aligned} |\,\psi\,\rangle &= (a\,|\,0\,\rangle + b\,|\,1\,\rangle) \times (c\,|\,0\,\rangle - d\,|\,1\,\rangle) \\ &= ac\,|\,00\,\rangle - ad\,|\,01\,\rangle + bc\,|\,10\,\rangle - bd\,|\,11\,\rangle \ . \end{aligned} \tag{14}$$

So, the probability of measuring Particle 1 in state $|\,0\,\rangle$ is $|\,\alpha\,|^2$, which is independent of information about Particle 2. Then, after measurement

$$|\,\psi\,\rangle = |\,0\,\rangle \times (c\,|\,0\,\rangle - d\,|\,1\,\rangle)\,. \tag{15}$$

Consider the following state, which is not separable.

$$|\,\psi\,\rangle = \frac{1}{\sqrt{2}}(|\,00\,\rangle + |\,11\,\rangle)\,. \tag{16}$$

Clearly, measuring either Particle 1 or Particle 2 will give information about the state of the other particle. Such states are said to be *entangled*.

$$P(A=0) = \tfrac{1}{2}\,, \quad P(B=1) = \tfrac{1}{2} \ . \tag{17}$$

Suppose that $B = 1$, then

$$P(A=0) = 0\,, \quad P(A=1) = 1\,. \tag{18}$$

Therefore, measuring the second bit alters the probability of the first Qbit

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4 How to compute with entanglement

Schrödinger's equation with turned on energy operator $\widehat{E}$.

Our first task is to build a NOT gate, which has the following effect:

$$|\,0\,\rangle \to |\,1\,\rangle \ , \quad |\,1\,\rangle \to |\,0\,\rangle \ . \tag{19}$$

And we have the matrix representations:

$$|\,0\,\rangle \sim \begin{pmatrix}1\\0\end{pmatrix} , \quad |\,1\,\rangle \sim \begin{pmatrix}0\\1\end{pmatrix} \ . \tag{20}$$

Then, U_{NOT} as a matrix is

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = -ie^{\frac{i\pi}{2}\left(\begin{smallmatrix}0 & 1\\1 & 0\end{smallmatrix}\right)}\,, \tag{21}$$

where the RHS represents the unitary transform from a magnetic field in the x direction, i.e., B_x, with $T\omega = \pi/2$, and where $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the Pauli matrix σ_x.

Example: Turn on B_y for time T.

$$|\,0\,\rangle \to \frac{1}{\sqrt{2}}(|\,0\,\rangle + |\,1\,\rangle) \quad \Big| \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{22a}$$

$$|\,1\,\rangle \to \frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle) \quad \Big| \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{22b}$$

Wow!

We have an operator U_H (U Hadamard) that converts the nonentangled states to entangled states. Note:

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) , \tag{23}$$

where the sigmas are Pauli matrices.

The next logical operator we will define is the so-called 'Controlled NOT' or CNOT operator. On the four pairs of 0's and 1's we can form, the following demonstrates what CNOT does to them:

$$\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{matrix} \quad \xrightarrow{\text{CNOT}} \quad \begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{matrix} \tag{24}$$

In summary, if the first bit is 0, it does nothing. But if the first bit is 1, it logically negates the second bit.

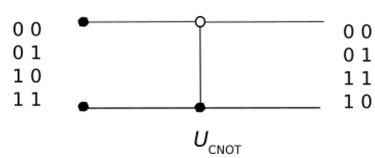

Figure 1. Here we diagram the action of U_{CNOT} on a pair of bits.

5 The No-Cloning Theorem

This is a quantum mechanical domain problem. Is it possible to take in a random binary state and duplicate it? Well, under Schrödinger evolution of the wave function, that process is both linear and unitary. And since the only operator we're allowed to employ on the evolution of states must be unitary, if we assume that there exists such an operator, but end up with a contradiction, then we will have proved that it is not possible to clone an arbitrary Qbit.

So, what we need to accomplish is to take some unknown Qbit $|\,x\,\rangle$ and some handy Qbit $|\,y\,\rangle$ to perform:

$$|\,x\,\rangle\,|\,y\,\rangle \xrightarrow{?} |\,x\,\rangle\,|\,x\,\rangle \ , \tag{25}$$

But

$$\alpha\,|\,x\,\rangle\,|\,z\,\rangle + \beta\,|\,x\,\rangle\,|\,y\,\rangle \longrightarrow (\alpha + \beta)\,|\,x\,\rangle\,|\,x\,\rangle \ , \tag{26}$$

where we have substituted in for both $|\,y\,\rangle$ and $|\,z\,\rangle$. However, normalization on the LHS requires that

$$|\,\alpha\,|^2 + |\,\beta\,|^2 = 1\,, \tag{27}$$

whereas normalization on the RHS requires that

$$|\,\alpha + \beta\,|^2 = 1\,, \tag{28}$$

and the two of these are not equal.

Exercise: Prove that there is no forgetting (use a similar argument).

6 Entangling Bits

We can entangle bits.

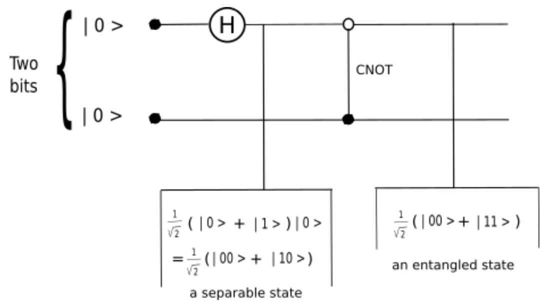

Figure 2. We've entangled two bits.

7 The Deutsch-Jozsa Algorithm

The Deutsch-Jozsa algorithm is a set of rules to guide a quantum system into the direction of performing an efficient computation.

As an example, consider the situation in which Matt knows a function that operates on a single bit, that is,

$$f : \{0,1\} \to \{0,1\}\,. \tag{29}$$

This will take two classical evaluations of f to determine if

$$f(0) = f(1)\,. \tag{30}$$

One simply evaluates f on both digits and then compare the results. However, if all we care about is whether or not $f(0) = f(1)$, what we really need to know is the value of $f(0) + f(1) \pmod 2$; for, if that value is either 0 or 2, then $f(0) = f(1)$. Otherwise, $f(0) + f(1) = 1$, which means that they're not equal to each other, but, instead, one is 0 and the other is 1. It turns out that we can use the clever manipulation of quantum bits to give the corect answer in one evaluation, that being of $f(0) + f(1)$.

So, we proceed in steps:

STEP 0: We do a preview or setup.

Setup: Matt provides a control bit.

$$U_f\ : |\,x\,\rangle\;|\,y\,\rangle \longrightarrow |\,x\,\rangle\;|\,f(x)\oplus y\,\rangle\ , \tag{31}$$

where $\oplus$ is addition mod 2.

STEP 1: Given $\psi = |\,0\,\rangle\;|\,1\,\rangle$, we Hardamard on both Qbits, to get

$$\begin{aligned}\psi' = H\psi &= \frac{1}{\sqrt{2}}[\;|\,0\,\rangle + |\,1\,\rangle] \times \frac{1}{\sqrt{2}}[|\,0\,\rangle - |\,1\,\rangle\;] \\ &= \frac{1}{2}[\,|\,0\,\rangle\;(|\,0\,\rangle - |\,1\,\rangle) + |\,1\,\rangle\;(|\,0\,\rangle - |\,1\,\rangle)\,]\,. \end{aligned}\tag{32}$$

STEP 2: Apply up

$$\begin{aligned}\psi' &= \frac{1}{2}\big[\;|\,0\,\rangle \times \big(\,|\,f(0)\,\rangle \oplus 0 - |\,f(0)\,\rangle \oplus 1\big) \\ &\qquad + |\,1\,\rangle \times \big(\,|\,f(1)\,\rangle \oplus 0 - |\,f(1)\,\rangle \oplus 1\big)\big] \\ &= \begin{cases} \frac{1}{2}[\,|\,0\,\rangle\;(0-1) + |\,1\,\rangle\;(0-1)\,] & \text{if } f(0)=0 \\ \frac{1}{2}[\,|\,0\,\rangle\;(1-0) + |\,1\,\rangle\;(1-0)\,] & \text{if } f(0)=1 \end{cases} \qquad (33)\\ &= \frac{1}{2}\big[\;|\,0\,\rangle\;(|\,f(0)\oplus 0\,\rangle - |\,f(0)\oplus 1\,\rangle) + |\,1\,\rangle\;(|\,f(1)\oplus 0\,\rangle - |\,f(1)\oplus 1\,\rangle)\,\big] \qquad (34)\\ &= \frac{1}{2}[\,(-1)^{f(0)}\,|\,0\,\rangle \times (|\,0\,\rangle - |\,1\,\rangle) + (-1)^{f(1)}\,|\,1\,\rangle \times (\,|\,0\,\rangle - |\,1\,\rangle)\,]\,. \qquad (35)\end{aligned}$$

Now, if $f(0) = f(1)$ (mod 2), then $f(0) + f(1) = 0$ (mod 2); otherwise, $f(0) + f(1) = 1$ (mod 2). Therefore,

$$\psi' = \tfrac{1}{2}(-1)^{f(0)}[\,|\,0\,\rangle + (-1)^{f(0)+f(1)}\,|\,1\,\rangle\;] \times (\,|\,0\,\rangle - |\,1\,\rangle)\,. \tag{36}$$

STEP 3: Forget about the second Qbit $(\,|\,0\,\rangle - |\,1\,\rangle)$.

If $f(0) = f(1)$ then

$$\psi' = \tfrac{1}{2}(-1)^{f(0)}[\,(|\,0\,\rangle + |\,1\,\rangle) \times (|\,0\,\rangle - |\,1\,\rangle)\,]\,, \tag{37}$$

and the first state $\sim (|\,0\,\rangle + |\,1\,\rangle)$.
Else if $f(0) \neq f(1)$ then

$$\psi' = \tfrac{1}{2}(-1)^{f(0)}[\,(|\,0\,\rangle - |\,1\,\rangle) \times (|\,0\,\rangle - |\,1\,\rangle)\,]\,, \tag{38}$$

and the first state $\sim (|\,0\,\rangle - |\,1\,\rangle)$.

STEP 4: Hadamard the first bit:[1]

$$\begin{aligned}\psi'' = H\psi' &= \frac{1}{2}[\,1 + (-1)^{f(0)\oplus f(1)})\,|\,0\,\rangle + \frac{1}{2}[\,1 - (-1)^{f(0)\oplus f(1)})\,|\,1\,\rangle\,) \qquad (39)\\ &= \begin{cases} |\,0\,\rangle + \frac{1}{2}\,0\;|\,1\,\rangle = |\,0\,\rangle\ , & \text{if } f(0)=1 \\ \frac{1}{2}\,0\;|\,0\,\rangle\,) + |\,1\,\rangle = |\,1\,\rangle & \text{if } f(0)=1 \end{cases}\,. \qquad (40)\end{aligned}$$

So, if we measure 0 we know they're the same; but if we measure 1, we know they're different.

[1] The Hadamard operation is an involution. Applying it will perform the inverse of what we saw in (22a) and (22b) to the effect of reverting the superpositions back to the singlet states $|\,0\,\rangle$ or $|\,1\,\rangle$.

STEP 5: This means for the first Qbit:

$$\begin{cases} 0 & \text{if they're the same}\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{41}$$

Therefore, this result occurs in half the number of steps than in the classical algorithm.

For N Qbits, we have 2^N possible states to deal with. If we imagine our previous f being defined on these N Qbits, given some problem to solve, we might have to evaluate all N Qbits. Anyway, we can try entangling the Qbits for better efficiency, in fact, this can be determined in a single measurement.

8 EPR

Suppose we begin with the entangled state

$$\psi = \frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)\,, \tag{42}$$

where these arrows are in the z direction. Let Alice take the first Qbit far away. Then let Bob take the second Qbit far away in the opposite direction. Now, the probability that Alice will measure her Qbit to be in the $+z$ direction is $1/2$. For the sake of example, say that she does measure $+1/2$, then she knows that Bob will measure $-1/2$ if he measures in the z-direction.

Question: How is this correlation established?

a) Either there exists some hidden variable that causally determines the correlated outcomes for both observers, or

b) there is non-locality (faster-than-light) influence afoot.

Now, if Alice measures $\uparrow_z$ and Bob measures $\uparrow_x$ and gets $\uparrow_x$////// but S_z and S_x do not commute. So, you can't have a state with definite S_z and S_x. According to Einstein, this is because quantum mechanics miust be incomplete.

9 Bell's Theorem

Let's begin with three binary properties A, B, C, which satify the following conditions:

$$N(A, \bar{B}) + N(B, \bar{C}) \geq N(A, \bar{C})\,, \tag{43}$$

where N stands for the count or number of, and the overbar represent logical negation. Also

$$N(A, \bar{B}, C) + N(\bar{A}, B, \bar{C}) \geq 0\,. \tag{44}$$

So far, so good. Now we apply these constraints to the EPR experiment in three trials.

I) Alice will measure up/down at angle $\theta = 0$, and Bob will measure at angle θ.

II) Then Alice will measure at angle θ, and Bob will measure at angle 2θ.

III) Then Alice will measure $\uparrow\ \downarrow$ at angle θ, and Bob will measure $\uparrow\ \downarrow$ at angle 2θ.

The values are:

$$A = \Big\uparrow_0\,, \quad B = \Big\uparrow_\theta\,, \quad \bar{B} = \Big\downarrow_\theta\,, \quad C = \Big\uparrow_{2\theta}\,, \quad \bar{C} = \Big\downarrow_{2\theta}\,. \tag{45}$$

Then

$$P(\uparrow_0,\downarrow_\theta)+P(\uparrow_\theta,\downarrow_{2\theta}) = LHS \tag{46}$$

$$P(\uparrow_0,\downarrow_{2\theta}) = RHS\,. \tag{47}$$

$$\uparrow_\theta = \cos\left(\tfrac{1}{2}\theta\right)\Big\downarrow_\theta + i\sin\left(\tfrac{1}{2}\theta\right)\Big\uparrow_\theta\,, \tag{48}$$

So,

$$P(\uparrow_0,\downarrow_\theta) = \sin^2(\tfrac{1}{2}\theta)\,, \tag{49}$$

$$P(\uparrow_\theta,\downarrow_{2\theta}) = \sin^2(\tfrac{1}{2}\theta)\,, \tag{50}$$

$$P(\uparrow_0,\downarrow_{2\theta}) = \sin^2\theta\,. \tag{51}$$

Notice that both (49) and (50) represent a rotation by θ. We have that

$$\Big\downarrow_\theta = \cos\left(\tfrac{1}{2}\theta\right)\Big\downarrow_0 + i\sin\left(\tfrac{1}{2}\theta\right)\Big\uparrow_0\,. \tag{52}$$

So, is

$$\sin^2(\tfrac{1}{2}\theta)+\sin^2(\tfrac{1}{2}\theta) \geq \sin^2\theta\,? \tag{53}$$

For $\theta << 1$ we have that:

$$(\tfrac{1}{2}\theta)^2+(\tfrac{1}{2}\theta)^2 \ngeq \theta^2\,. \tag{54}$$

The prediction of this calculation is a violation of Bell's Inequality. And this implies that there are no classical hidden variables that can account for EPR correlation.

By the way, Bell's Theorem was tested by Alain Aspect — and QM won.