Spin-half formalism for quantum mechanics

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Abstract

This paper contains my notes on how to develop the quantum mechanical equations for spin-half behavior in magnetic fields. This presentation uses my notes on the lecture from Barton Zwiebach[1]. The fault for any inaccuracies in this presentation is strictly my own.

1 Linear algebra review

This paper assumes that the reader is already familiar with the formalism of the quantum mechanics of Schrödinger, which requires knowledge of operators, vectors in Hilbert space, eigenvalues and eigenfunctions.

When using the Schrödinger equation with potential given, we solve for the possible eigenstates and we know the probabilities of those states. At measurement, these states are mutually exclusive. To be in one eigenstate is definitely not to be simultaneously in any other state. Mathematically, we say that these distinct states are orthogonal.

The Schrödinger equation usually allows for an infinite number of eigenfunctions and eigenstates. When we deal with the experimental spin states of, say, an silver atom exiting a Stern-Gerlach apparatus, we will be interested in only two possible states: either state 'up' or state 'down', which corresponds to the direction of the magnetic field and opposite to it.

In the Schrödinger formalism, we have operators that operate on wave functions. In this new formalism, we will also have operators that operate on two possible states. In the spin formalism, we will be respresenting these two states as 2×1 matrices. So, by analogy with the Schrödinger formalism, what would be the natural form for operators on these 2×1 matrices? Of course, they would be 2×2 matrices.

We will need to know how to find eigenvectors and eigenvalues of a given 2×2 matrix. Let M be such a matrix. Further, let v be a 2×1 matrix (vector) such that for real number λ,[2]

$$Mv = \lambda v \,. \tag{1}$$

That is, the operation of M on vector v has the effect of leaving its direction fixed, but rescaling it to the value λv. The vector v is said to be an eigenvector of the matrix M, and the scalar λ is said to be the eigenvalue corresponding to eigenvector v.

The usual way to proceed to find the eigenvectors and eigenvaues is to solve the following equation for λ:

$$\det(M - \lambda \mathbf{I}) = 0 \,. \tag{2}$$

If we use numbers for the components of M, we have

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,, \tag{3}$$

[1]MIT 8.05 Quantum Physics II, Fall 2013, 4. Spin One-half, Bras, Kets, and Operators.

[2]For the sake of doing quantum mechanics, we require that our matrix operators are hermitian, and that entails that eigenvalues are real.

and

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

is the unit matrix in the algebra of 2×2 matrices.

Then (2) becomes

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0\,, \tag{5}$$

On multiplying this out, we have

$$\lambda^2 - \lambda(a+d) + (ad - bc) = 0\,. \tag{6}$$

Since this is a quadric equation, it's guaranteed to have two roots for λ. However, it's sometimes useful to rewrite this last equation as

$$\lambda^2 - \lambda \mathrm{Tr}\,(M) + \det M = 0\,. \tag{7}$$

where $\mathrm{Tr}\,(M) = a + d$, that is, it's the sum of the components on the main diagonal, which is called the *trace* of the matrix. And $\det M = ad - cb$ is called the determinant of M.[3]

2 Spin Eigenstates and Eigenvectors

By convention, we define the 'up' eigenstate as along the $+z$ axis, and the down eigenstate as along the $-z$ axis, and they are, respectively, represented by

$$|\,z;+\,\rangle \quad \text{and} \quad |\,z;-\,\rangle\;, \tag{8}$$

and we have the relations, among others:

$$\langle\,z;-\,|\,z;+\,\rangle = 0\,, \qquad \langle\,z;+\,|\,z;+\,\rangle = 1\,, \qquad \text{etc.} \tag{9}$$

The corresponding angular momentum values to these vectors are

$$S_z = \frac{\hbar}{2} \quad \text{and} \quad S_z = -\frac{\hbar}{2}\,. \tag{10}$$

Since we have eigenvalues that correspond to spin eigenvalues, if we are to develop this area of quantum physics in analogy with that of the Schrödinger formalism, we need to invent a spin operator $\widehat{S}_z$ that correspond to these eigenvalues, such that

$$\widehat{S}_z\,|\,z;+\,\rangle = \frac{\hbar}{2}\,|\,z;+\,\rangle\;, \tag{11}$$

$$\widehat{S}_z\,|\,z;-\,\rangle = -\frac{\hbar}{2}\,|\,z;-\,\rangle\;. \tag{12}$$

Now, we attempt to represent $|\,z;+\,\rangle$ and $|\,z;-\,\rangle$, respectively, by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{13}$$

One reason this representation makes sense is, not only do we have discrete states to deal with, but only two of them. The result of a quantum measurement on a particle with spin will give us the value of either up or down. Thus, the two states must be mutually exclusive, or orthogonal to each

[3] The determinant is defined on all square matrices, but for dimensions higher than 2, it gets complicated fast.

other (their inner product is zero). We can test this right now. Two states $|\,\psi\,\rangle$ and $|\,\varphi\,\rangle$ are said to be orthogonal to each other if

$$\langle\,\psi\mid\varphi\,\rangle = 0\,. \tag{14}$$

In the case of our two vectors, we have

$$\begin{pmatrix}1\\0\end{pmatrix}^\dagger \begin{pmatrix}0\\1\end{pmatrix} = (1\ 0)^* \begin{pmatrix}0\\1\end{pmatrix} = 0\,, \tag{15}$$

where the asterisk represents complex conjugation. So far, so good.

The amazing thing is that just these two vectors will constitute a basis for our entire spin state collection, which includes for spin probabilities along both the x-axis and the y-axis, but the proof of this will take some time. Anyway, let's take this claim seriously and claim that an arbitrary state vector has the form

$$|\,\psi\,\rangle = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}\,, \tag{16}$$

where c_1 and c_2 are arbitrary complex numbers. All we have to do now is to prove it!

It's time now for some notational simplifications. Let's begin with the replacements

$$|\,1\,\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |\,2\,\rangle = \begin{pmatrix}0\\1\end{pmatrix}\,. \tag{17}$$

Then

$$\langle\, i\mid j\,\rangle = \delta_{ij}\,. \tag{18}$$

Now, we define two arbitrary ket vectors as follows:

$$|\,\alpha\,\rangle = \alpha_1\,|\,1\,\rangle + \alpha_2\,|\,2\,\rangle\,, \tag{19}$$

$$|\,\beta\,\rangle = \beta_1\,|\,1\,\rangle + \beta_2\,|\,2\,\rangle\,. \tag{20}$$

Next, we transform each ket into a bra vector by hermitian conjugation, as follows:

$$\langle\,\alpha\,| = \alpha_1^*\,\langle\,1\,| + \alpha_2^*\,\langle\,2\,|\,, \tag{21}$$

$$\langle\,\beta\,| = \beta_1^*\,\langle\,1\,| + \beta_2^*\,\langle\,2\,|\,. \tag{22}$$

And the reason we did this is because we have to take inner products of these vectors at times. Let's show it for $|\,\alpha\,\rangle$ and $|\,\beta\,\rangle$

$$\langle\,\alpha\mid\beta\,\rangle = (\alpha_1^*, \alpha_2^*)\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \alpha_1^*\beta_1 + \alpha_2^*\beta_2\,. \tag{23}$$

Obviously, $\langle\,\alpha\mid\alpha\,\rangle = \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2$ is a nonnegative real number, and when this number is not zero, we use this fact to normalize $|\,\alpha\,\rangle$.

Okay, now that we have the state vector to operate on, it's obvious what we should use as the operators on these vectors: 2×2 matrices. For example, we can set

$$\widehat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\,. \tag{24}$$

We're supposed to have that

$$\widehat{S}_z\,|\,1\,\rangle = +\frac{\hbar}{2}\,|\,1\,\rangle \quad \text{and} \quad \widehat{S}_z\,|\,2\,\rangle = -\frac{\hbar}{2}\,|\,2\,\rangle\,. \tag{25}$$

So, let's try it.

$$\widehat{S}_z\,|\,1\,\rangle = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}\,, \tag{26}$$

which gives us the spin-up eigenstate with angular momentum $\hbar/2$. And for the other basis vector:

$$\widehat{S}_z \,|\, 2\,\rangle = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\ 1\end{pmatrix}\,, \tag{27}$$

which gives us the spin-down eigenstate with angular momentum $-\hbar/2$.

Now we come to the big question. If we orient the Stern-Gerlach apparatus in a direction other than the $\pm z$ directions: a) what will happen, and b) can we represent the eigenstates with the same two vectors, or will we need to come up with additional basis vectors? The answer to the first question seems obvious enough. The behavior of the particles will be similar to whatever happens in the z direction, that being that there will be only two possible outcomes, along or anti-along the direction of the external magnetic field.

But the answer to the second question seems easy to grasp — once you hear it. We will proceed in analogy to the case of atomic orbital angular momentum relations. If we don't make the assumption that spin angular momentum has to follow the same patterns as orbital angular momentum then we're pretty much stuck where we are. Anyway, we'll take this reasonable tack and see how far we get with it.

Let's begin with the relation for orbital angular momentum,

$$\widehat{L}_z = \widehat{x}p_y - \widehat{y}p_x\,, \tag{28}$$

and we note that $\widehat{L}_z$ is hermitian. If we make the identifications:

$$\widehat{L}_x = \widehat{L}_1\,, \quad \widehat{L}_y = \widehat{L}_2\,, \quad \widehat{L}_z = \widehat{L}_3\,, \tag{29}$$

then we get this simplifying relations

$$[\,\widehat{L}_i, \widehat{L}_j\,] = i\hbar\epsilon_{ijk}\widehat{L}_k\,. \tag{30}$$

From this we get that:

$$\begin{aligned}[\,\widehat{L}_x, \widehat{L}_y\,] &= i\hbar\widehat{L}_z\,,\\ [\,\widehat{L}_y, \widehat{L}_z\,] &= i\hbar\widehat{L}_x\,,\\ [\,\widehat{L}_z, \widehat{L}_x\,] &= i\hbar\widehat{L}_y\,.\end{aligned} \tag{31}$$

So, let's just jump right in and suppose analogous relationships for spin angular momentum:

$$\begin{aligned}[\,\widehat{S}_x, \widehat{S}_y\,] &= i\hbar\widehat{S}_z\,,\\ [\,\widehat{S}_y, \widehat{S}_z\,] &= i\hbar\widehat{S}_x\,,\\ [\,\widehat{S}_z, \widehat{S}_x\,] &= i\hbar\widehat{S}_y\,.\end{aligned} \tag{32}$$

By analogy with (30), we expect that,

$$[\,\widehat{S}_i, \widehat{S}_j\,] = i\hbar\epsilon_{ijk}\widehat{S}_k\,. \tag{33}$$

Next, we'll require hermiticity. For a 2×2 matrix with components a, b, c, d being real numbers, then

$$M = \begin{pmatrix}2c & a-ib\\ a+ib & 2d\end{pmatrix} \tag{34}$$

is hermitian. Clearly this matrix is invariant under the operation of taking its complex transpose. And if we subtract from M all matrices of the form $(c+d)\mathbf{I}$, where $\mathbf{I}$ is the identity matrix for the space of 2×2 matrices, then the commutation relations will not be affected.

Let's show that if we have two operators A and B that we can replace them by two other operators A' and B' defined by

$$A' = A - \alpha\mathbf{I} \qquad \text{where } \alpha \text{ is an abitrary complex number.} \tag{35}$$

and similarly for operator B. For our purposes here, A and B are just arbitrary square matrices. Next, we show that their commutators are unaffected:

$$\begin{aligned} [\,A', B'\,] &= (A - \alpha\mathbf{I})(B - \alpha\mathbf{I}) - (B - \alpha\mathbf{I})(A - \alpha\mathbf{I}) \\ &= AB - \alpha(A + B) - \alpha^2\mathbf{I} - [\,BA - \alpha(A + B) - \alpha^2\mathbf{I}\,] \\ &= AB - BA = [\,A, B\,]\,. \end{aligned} \tag{36}$$

Our matrix found in (34) now takes the form (of a traceless matrix):

$$\begin{pmatrix} c - d & a - ib \\ a + ib & d - c \end{pmatrix} . \tag{37}$$

In our construction of $\widehat{S}_x$ and $\widehat{S}_y$, we want to be sure not to include multiples of $\widehat{S}_z$, which directs us to eliminate matrices in the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} = \alpha\sigma_z\,. \tag{38}$$

This new requirement forces us to restrict our attentions to matrices of the form

$$\begin{pmatrix} 0 & a - ib \\ a + ib & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{39}$$

which is, nevertheless, the sum of hermitian matrices. Thus, it appears that we have found a 4-dimensional linear space with basis vectors

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{40}$$

where the sigma matrices are known as the *Pauli Matrices*.

Now I want to prove that these vectors do indeed form a 4-D linear space by proving that no one of them can be written as a sum of the other three of them. To establish this proof, I need the facts that 1) each of these 2×2 matrices squares to the identity matrix, 2) that the trace of all but the identity matrix is zero. Of course, the trace of the identity matrix is 2. I'll leave the proof of that to the reader, as well as the proof that the product of any two of $\sigma_x, \sigma_y, \sigma_z$ is a scalar times the third one. Now, if we form the equation

$$a\mathbf{I} + b\sigma_x + c\sigma_y + d\sigma_z = 0\,, \tag{41}$$

we can show that the space is these for vectors span the space as a basis (a minimal spanning set) if we can show the only solution for these coefficients is zero for each of them. What if we take the trace across (41)? What we get is the $2a = 0$ so that $a = 0$, though b, c, d are still to be determined. But can we express one of the sigma matrices in terms of the others? Let's try.

$$b\sigma_x + c\sigma_y = -d\sigma_z\,. \tag{42}$$

Next, multiply through by σ_z on the right to get

$$-ib\sigma_y + ic\sigma_x = -d\mathbf{I}\,. \tag{43}$$

Now if we take the trace across this equation, we get that $0 = -2d$, and thus $d = 0$. So, if we continue this procedure, we can show that b and c must also be zero.

Now, it's time to get to the business of finding eigenvectors and eigenvalues for $\widehat{S}_x$ and $\widehat{S}_y$. So,

$$|\,\sigma_x - \lambda \mathbf{I}\,| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0\,, \tag{44}$$

from which we have that

$$\lambda_{\pm} = \pm 1\,. \tag{45}$$

Setting up the equation the eigenvectors,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = +1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\,, \tag{46}$$

we can solve for the components of vector $\mathbf{v}$ by inspection to get (with proper normalization included)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,. \tag{47}$$

Next, for the equation of the eigenvector to the negative eigenvalue,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\,. \tag{48}$$

We can solve for the components of vector $\mathbf{v}$ by inspection to get (with proper normalization included)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,. \tag{49}$$

Similarly, for $\widehat{S}_y$, we get eigenvectors, starting with eigenvalue $+1$:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}\,, \tag{50}$$

and with eigenvalue -1:

$$v' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}\,. \tag{51}$$

And, as a test:

$$\langle\, v \mid v' \,\rangle = 0\,. \tag{52}$$

So, what we have shown is that by using complex entries in 2×1 matrices, we have enough freedom to find eigenvectors for the three directions in space, with their two orientations each, all of them as linear combinations of just $|\,z; +\,\rangle$ and $|\,z; -\,\rangle$.

Next, let's check their commutation relations. A typical relation is:

$$\begin{aligned} [\,\widehat{S}_x, \widehat{S}_y\,] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\ &= \frac{\hbar^2}{4} \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\ &= i\hbar \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar \widehat{S}_z\,. \end{aligned} \tag{53}$$

If we make the identifications,

$$\widehat{S}_x = \widehat{S}_1\,, \quad \widehat{S}_y = \widehat{S}_2\,, \quad \widehat{S}_z = \widehat{S}_3\,, \tag{54}$$

then we can say that

$$\widehat{S}_i = \frac{\hbar}{2}\sigma_i \qquad (i = 1, 2, 3)\,. \tag{55}$$

Beginning with

$$\widehat{S}_x \,|\, x; \pm \rangle = \pm \,|\, x; \pm \rangle\ , \tag{56}$$

we can write

$$|\, x; + \rangle = \frac{1}{\sqrt{2}}\left(|\, z; + \rangle + |\, z; - \rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\ , \tag{57a}$$

$$|\, x; - \rangle = \frac{1}{\sqrt{2}}\left(|\, z; + \rangle - |\, z; - \rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\ . \tag{57b}$$

So, surprise! The $|\, x; \pm \rangle$ are just linear combinations of our original eigenvectors $|\, z; \pm \rangle$.

We can invert Eqs. (57a) and (57b), to get

$$|\, z; + \rangle = \frac{1}{\sqrt{2}}\left(|\, x; + \rangle + |\, x; - \rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\ , \tag{58a}$$

$$|\, z; - \rangle = \frac{1}{\sqrt{2}}\left(|\, x; + \rangle - |\, x; - \rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\ . \tag{58b}$$

So, if we prepare a particle in a $+z$ state and then send it into a $+x$ filter [i.e., force it to choose to output the particle into either a plus or minus x direction of spin], what's the probability of it exiting in the $+x$ direction?

$$\begin{aligned}\langle\, x; + \mid z; + \,\rangle &= \tfrac{1}{2}\,\langle\, x; + \,|\, \big(\,|\, x; + \rangle + |\, x; - \rangle\big) \\ &= \tfrac{1}{2}\,\langle\, x; + \mid x; + \,\rangle = \tfrac{1}{2}\,. \end{aligned} \tag{59}$$

For completeness,

$$\begin{aligned}\widehat{S}_y \,|\, y; \pm \rangle &= \pm\frac{\hbar}{2}\,|\, y; \pm \rangle\ , \\ |\, y; \pm \rangle &= \frac{1}{\sqrt{2}}\left(|\, z; + \rangle \pm i\,|\, z; - \rangle\right)\ . \end{aligned} \tag{60}$$

3 Arbitrary directions in space

Now that we have a basis of directions, can we meaningfully talk about arbitrary spin directions in space?

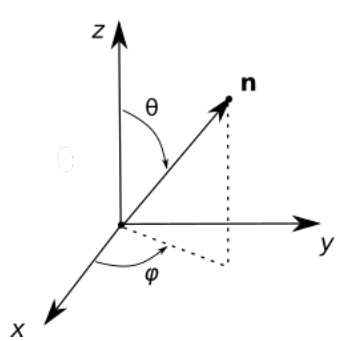

Figure 1. Vector **n** represents an arbitrary spin direction (in physical space) of unit length. The coordinate system that uses **n**, θ, and φ is called 'spherical'.

Referencing Fig. 1, we can claim a form for **n** in rectangular coordinates as:

$$\mathbf{n} = n_x \mathbf{e}_x + n_y \mathbf{e}_y + n_z \mathbf{e}_z \,, \tag{61}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors along their respective x, y, z directions, and $\mathbf{n}^2 = 1$. Now we can write an operator **S** as a sum of the previously defined spin operators by

$$\begin{aligned} \mathbf{S} &= (\widehat{S}_x, \widehat{S}_y, \widehat{S}_z) \\ &= \widehat{S}_x \mathbf{e}_x + \widehat{S}_y \mathbf{e}_y + \widehat{S}_z \mathbf{e}_z \,. \end{aligned} \tag{62}$$

Next, we can define the operator:

$$\widehat{S}_{\mathbf{n}} = \mathbf{n} \cdot \mathbf{S} = n_x \widehat{S}_x + n_y \widehat{S}_y + n_z \widehat{S}_z \,. \tag{63}$$

The following are the standard relations between the components of **n** and the angles θ, and φ:

$$n_x = \cos\varphi \sin\theta \,, \tag{64a}$$

$$n_y = \sin\varphi \sin\theta \,, \tag{64b}$$

$$n_z = \cos\theta \,. \tag{64c}$$

Now, a return to (63) and we use the Pauli matrices:

$$\begin{aligned} \widehat{S}_{\mathbf{n}} &= \frac{\hbar}{2}(n_x \sigma_1 + n_y \sigma_2 + n_z \sigma_3) \\ &= \frac{\hbar}{2} \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} . \end{aligned} \tag{65}$$

Eigenvalues — Eigenvectors of $\widehat{S}_{\mathbf{n}}$: For these, we need, as predicatable, the secular equation[4]

$$\left| \widehat{S}_{\mathbf{n}} - \lambda \mathbf{I} \right| = 0 \,, \tag{66}$$

or,

$$\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta e^{-i\varphi} \\ \frac{\hbar}{2}\sin\theta e^{i\varphi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = 0 \,. \tag{67}$$

Solving for the eigenvalues, we have that

$$\lambda_{\pm} = \pm \frac{\hbar}{2} \,. \tag{68}$$

The corresponding eigenvectors $\mathbf{v}_{\pm}$ satisfy the equation

$$(\widehat{S}_{\mathbf{n}} - \lambda_{\pm} \mathbf{I}) \mathbf{v}_{\pm} = 0 \,, \tag{69}$$

[4] Also called the *characteristic equation*. It is the polynomial equation one gets after expanding the determinant equation.

where $\widehat{S}_{\mathbf{n}}$ satisfies

$$\widehat{S}_{\mathbf{n}} \,|\, \mathbf{n}; \pm \,\rangle = \frac{\hbar}{2} \,|\, \mathbf{n}; \pm \,\rangle \,. \tag{70}$$

Let's represent $|\, \mathbf{n}; \pm \,\rangle$ as a linear combination of the z states:

$$|\, \mathbf{n}; \pm \,\rangle = c_1 \,|\, z; + \,\rangle + c_2 \,|\, z; - \,\rangle = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \,. \tag{71}$$

Thus,

$$(\widehat{S}_{\mathbf{n}} - \lambda_{\pm} \mathbf{I}) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \,, \tag{72}$$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \, e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -(\cos\theta + 1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \,. \tag{73}$$

If we choose the values $c_1 = \sin\theta e^{-i\varphi}$ and $c_2 = -(\cos\theta - 1)$, that will solve the top-line equation. We easily then find that it also satisfies the bottom-line equation. So, we have tentative components. Nevertheless, let's now normalize them. This process begins with a trigonometric substitution:

$$c_2 = e^{i\varphi} \frac{1 - \cos\theta}{\sin\theta} c_1 = e^{i\varphi} \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} c_1 \,. \tag{74}$$

Warning: Since (73) is homogeneous, solutions on c_1 and c_2 are not uniquely determined. Therefore, we may have to modify c_1 and c_2 to conform to additional constraints on the problem.

One constraint these components need to conform to is

$$|\, c_1 \,|^2 + |\, c_2 \,|^2 = 1 \,. \tag{75}$$

Using the value of c_2 in terms of c_1 we just calculated, we get

$$|\, c_1 \,|^2 + \left| \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} c_1 \right|^2 = 1 \,, \tag{76}$$

which yields

$$|\, c_1 \,|^2 = \cos^2 \tfrac{1}{2}\theta \,. \tag{77}$$

On choosing the positive square root for c_1, we can write,

$$c_1 = \cos \tfrac{1}{2}\theta \qquad \text{and} \qquad c_2 = \sin \tfrac{1}{2}\theta \, e^{i\varphi} \,. \tag{78}$$

Therefore,

$$|\, \mathbf{n}; + \,\rangle = \cos \tfrac{1}{2}\theta \,|\, z; + \,\rangle + \sin \tfrac{1}{2}\theta e^{i\varphi} \,|\, z; - \,\rangle \,. \tag{79}$$

Going through a similar process to arrive at the coefficients for the λ_- case, we get the tentative result (it will have to be tested against previous results).

$$|\, \mathbf{n}; - \,\rangle = \sin \tfrac{1}{2}\theta \,|\, z; + \,\rangle - \cos \tfrac{1}{2}\theta e^{i\varphi} \,|\, z; - \,\rangle \,. \tag{80}$$

We're about to find out that our solution isn't quite right. Let's take this general solution and apply it to the case $\mathbf{n} = \hat{\mathbf{z}}$: where $\theta = 0$ and φ is undetermined

$$|\, \hat{\mathbf{z}}; - \,\rangle = 0 \,|\, z; + \,\rangle - e^{i\varphi} \,|\, z; - \,\rangle \,, \tag{81}$$

So, let's move the phase factor from the second term to the first term, to get

$$|\, \mathbf{n}; - \,\rangle = -\sin \tfrac{1}{2}\theta e^{-i\varphi} \,|\, z; + \,\rangle + \cos \tfrac{1}{2}\theta \,|\, z; - \,\rangle \,. \tag{82}$$

And thus we have shown that a spin operator in an arbitrary direction in space has been provided for the as a linear combination of just two basis vectors $|\, z; \pm \,\rangle$.