Lecture 7: Interacting Theories and the S-Matrix

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Abstract

This presentation is my read-along notes on the Lecture 7 from Hong Liu: MIT 8.323 Relativistic Quantum Field Theory I, Spring 2023 The fault for any inaccuracies in this presentation is strictly my own.

1 Some review

Green's function

$$G_+(x,x') \equiv \langle\, x \mid x'\,\rangle = \langle\, 0 \,|\, \phi(x)\phi(x') \mid 0\rangle \ . \tag{1}$$

(1) Heuristically we interpret this as a transition amplitude for a particle going from x' to x.

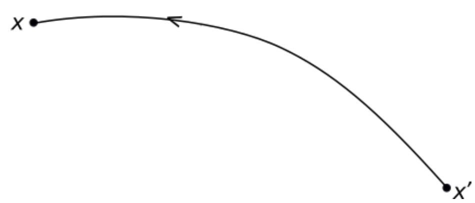

Figure 1.

(2) The correlation function describes the correlation between $\phi(x')$ and $\phi(x)$.

Similarly,

$$G_-(x,x') \equiv \langle\, 0 \,|\, \phi(x')\phi(x) \mid 0\rangle \ . \tag{2}$$

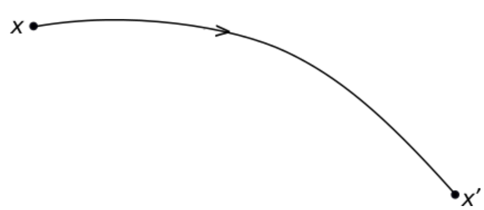

Figure 2.

Then for the Feynman propagator,

$$G_F(x,x') = \theta(t-t')G_+(x,x') + \theta(t'-t)G_-(x,x')\,. \tag{3}$$

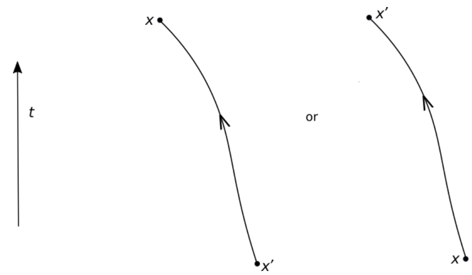

Figure 3.

Time-ordered correlation $= G_F$

$\phi(x')\,|\,0\,\rangle$ generates a particle;

$\langle\,0\,|\,\phi(x)$ generates an antiparticle.

$$G_{R,A,F}(x,x') = -i\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x,x')}}{k^2+m^2} \quad (\text{retarded, advanced})\,, \tag{4}$$

where $k^2+m^2 = -\omega^2+\omega_{\mathbf{k}}^2$. Thus we have poles at $\omega = \pm\omega_{\mathbf{k}}$.

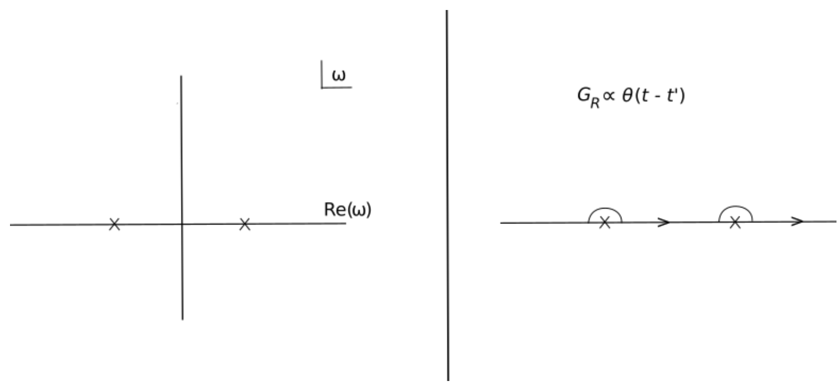

Figure 4.

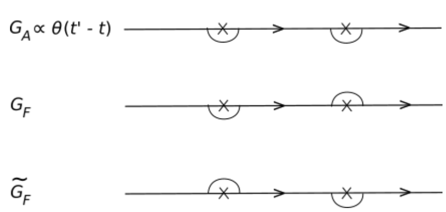

Figure 5.

Physics Tricks

Nice trick!

$$G_{R,A}(x,x') = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{ik(x,x')}}{-(\omega+i\epsilon)^2+\omega_{\mathbf{k}}^2} \qquad (\epsilon << 1)\,, \qquad \omega = \pm\omega_k \pm i\epsilon\,,$$
$$G_F = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{ik\cdot(x-x')}}{k^2+m^2-i\epsilon} \qquad (\epsilon \to 0)\,. \tag{5}$$

Final remarks:

(1)

G_+, G_R have applications in CM (condensed matter).

G_F has a role in the next chapter.

(2) More general than between two points is:

$$\langle\, \psi \mid \phi(x_1)\phi(x_2)\cdots\phi(x_n) \mid \psi\,\rangle \tag{6}$$

The general state built from a and $a^\dagger$. This theory is fully solved.

Fact:

$$\langle\, 0 \mid \phi(x_1)\phi(x_2)\cdots\phi(x_n) \mid 0\,\rangle \\ \to \text{factorize into sums of products of 2-pt functions.} \tag{7}$$

Note: a and $a^\dagger$ must be paired so that a creation is followed by an annihilation.

Example. 4-point function.

$$\begin{aligned}\langle\, 0 \mid \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \mid 0\,\rangle \\ &\to G_+(x_1,x_2)G_+(x_3,x_4) \\ &\quad + G_+(x_1,x_4)G_+(x_2,x_3) \\ &\quad + G_+(x_1,x_3)G_+(x_2,x_4)\end{aligned} \tag{8}$$

for all possible pairings.

(3) Besides ϕ, we can have "composite" operators, such as

$$\phi^2(x), \phi^3(x), \mathcal{H}, T^{\mu\nu}\,, \tag{9}$$

where $\mathcal{H}$ is the Hamiltonian and $T^{\mu\nu}$ is the stress tensor.

This will cause singularities at points, requiring renormalization. The constant burden of QFT is to find good ways to renormalize singularities.

2 Chapter 3: Interacting Theories

The path-integral approach to Action.

Previously:

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2\,, \tag{10}$$

for free particles, and we add to this the term $-\dfrac{\lambda}{4!}\phi^4$.

$$\mathscr{L} = \mathscr{L}_0 - \frac{\lambda}{4!}\phi^4 \,, \tag{11}$$

where λ parameterizes the new Lagrangian, where $\mathcal{L}_0$ is the Lagrangian for the free particle.

Equation of motion:

$$\partial^2\phi - m^2\phi - \frac{\lambda}{6}\phi^3 = 0 \,, \tag{12}$$

which has no exact solution. This is an interaction term ϕ^3. To solve (12) approximately, treat λ as a small parameter.

Experimentally, particle interactions are tested by particle scatterings.

DIS – Deep Inelastic Scattering (quarks).

One of the key elements of scattering is the S-matrix. $S_{\beta\alpha}$ serves as a matrix element of the evaluation operator.

3 Mathematical idealization

1) At $t = -\infty$: prepare initial state as localized wave packet infinitely far apart from target.

2) Incoming particle scatters off target particles.

3) $t = +\infty$: final particles end up infinitely far apart.

By consequence we can neglect interaction everywhere except when they are very close. In other words, the initial state is of free particles and the final state is as well, at which point particle identities are well defined.

Denote

$$\alpha = \{p_1, \ldots, p_k\} \,, \tag{13}$$

which represents the initial particle momenta.

$$\beta = \{p'_1, \ldots, p'_n\} \,, \tag{14}$$

states, where $\alpha \to \beta$. In the Heisenberg picture

$$\langle\, \beta \mid U(+\infty, -\infty)|\alpha\,\rangle \;, \tag{15}$$

where $U(+\infty, -\infty)$ is the *evolution operator*. In the Schrödinger picture, we have

$$\langle\, \beta, +\infty \mid \alpha, -\infty\,\rangle \equiv S_{\beta\alpha} \,, \tag{16}$$

which is the S-matrix, which can, in principle, be infinite.

For weak interactions, we have that

$$S = \mathbb{1} + iT \,, \tag{17}$$

where T captures the interaction effects and

$$S = \mathbb{1} \tag{18}$$

is the no-interaction operator and is the same as $\delta_{\alpha\beta}$.

Due to translation symmetry, we expect energy-momentum conservation.

$$S_{\beta\alpha} = \delta_{\beta\alpha} + iT_{\beta\alpha}\,. \tag{19}$$

To simplify,

$$iT_{\beta\alpha} = i(2\pi)^4\delta^{(4)}(p_\alpha - p_\beta)M_{\alpha\to\beta}\,, \tag{20}$$

where $M_{\alpha\to\beta}$ is called the *scattering amplitutde.*

S is the matrix with properties:

1) $U(+\infty, -\infty)$ is unitary, S is a unitary matrix.

2) For any symmetry Λ of the Hamiltonian, $[\,\Lambda, H\,] = 0$ and $\Lambda\Lambda^\dagger = 1$.

When Λ is an operator for some symmetry, (where Λ commutes with U)

$$\Lambda = e^{i\omega M}\,, \tag{21}$$

then

$$S_{\Lambda\beta,\Lambda\alpha} = S_{\beta,\alpha}\,. \tag{22}$$

Proof: Start with

$$S_{\Lambda\beta,\Lambda\alpha} = \delta_{\Lambda\beta,\Lambda\alpha} + i(2\pi)^4\delta^{(4)}(p_{\Lambda\alpha} - p_{\Lambda\alpha})M_{\Lambda\alpha\to\Lambda\beta}\,. \tag{23}$$

We wish to calculate $M_{\Lambda\alpha\to\Lambda\beta}$ scattering amplitude.

4) The LSZ Theorem

$M_{\Lambda\alpha\to\Lambda\beta}$ can be obtained from

$$\langle\,\Omega\mid T\phi(x_1)\cdots\phi(x_n)\,|\,\Omega\,\rangle\;, \tag{24}$$

where $\phi(x_1)\cdots\phi(x_n)$ are listed in reverse time order, and Ω is the vacuum state.