Lecture 10:Time-Ordered Correlation Functions in QFT

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Abstract

This presentation is my read-along notes on the Lecture 10 from Hong Liu: MIT 8.323 Relativistic Quantum Field Theory I, Spring 2023. The fault for any inaccuracies in this presentation is strictly my own.

1 Some review

Remember from last time we have the single-particle propagator

$$G_n = \langle\, 0 \,|\, T(\widehat{x}(t_1) \cdots \widehat{x}(t_n)) \,|\, 0 \,\rangle\,. \tag{1}$$

with

$$L = \tfrac{1}{2} m \dot{x}^2 - V(x)\,. \tag{2}$$

Let

$$X \equiv \widehat{x}(t_1) \cdots \widehat{x}(t_n)\,, \tag{3}$$

then

$$G_n = \frac{\int Dx(t) X e^{iS_\epsilon[x(t)]}}{\int Dx(t) e^{iS_\epsilon[x(t)]}}\,, \tag{4}$$

with boundary conditions

$$x(+\infty) = x(-\infty) = 0\,, \tag{5}$$

which are the standard boundary conditions. With this we have $H \to H(1 - i\epsilon)$. Then, after G_n is calculated, we take $\epsilon \to 0$.

The generating functional

$$z_n = \int dx e^{i\lambda f(x)} x^n\,. \tag{6}$$

We have the more convenient integral

$$z(a) = \int dx\, e^{i\lambda f(x) + ixa}\,, \tag{7}$$

$$z_n = \frac{1}{i^n} \frac{\partial^n z(a)}{\partial a^n}\Big|_{a=0}\,. \tag{8}$$

The derivatives are easier to work with than the integrals.

$$z(a) = \sum_{n=0}^{\infty} \frac{i^n}{n!} z_n a^n\,. \tag{9}$$

We call $z(a)$ the generating function.

Let $J(t)$ be the analog of a

$$z[J(t)] = \int Dx(t)\, e^{iS[x(t)]+i\int dtJ(t)x(t)]} \,. \tag{10}$$

The functional derivative:

$$\frac{\delta J(t')}{\delta J(t)} = \delta(t-t')\,, \tag{11}$$

$$\frac{\delta z[J]}{\delta J(t)} = i\int Dx\, x(t) e^{iS+i\int Jx} \,. \tag{12}$$

Introduce

$$z_0 \equiv z[J=0] = i\int Dx\, e^{iS} \,. \tag{13}$$

Then, for the one-point function:

$$\langle\, 0\,|\,\widehat{x}(t)\,|\,0\,\rangle = \frac{1}{i}\frac{1}{z_0}\frac{\delta z[J]}{\delta J(t)}\bigg|_{J(t)=0} \,. \tag{14}$$

So, generalizing,

$$G_n = \frac{1}{i^n}\frac{1}{z_0}\frac{\delta^n z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\bigg|_{J(t)=0} \,. \tag{15}$$

Do the path integral once and then do the derivatives.

$$\frac{z[J]}{z_0} = \langle\, 0\,|\,\widehat{x}\,|\,0\,\rangle = \frac{1}{i}\frac{1}{z_0}\frac{\delta z[J]}{\delta J(t)}\bigg|_{J(t)=0} \,. \tag{16}$$

$$\frac{z[J]}{z_0} = \langle\, 0\,|\,T(e^{i\int dtJ(t)x(t)})\,|\,0\,\rangle \,. \tag{17}$$

So, how does this work? In (4), X can be anything. Comparing to $z[J(t)]$, set

$$e^{iS[x(t)]+i\int dtJ(t)x(t)]} \longrightarrow X\, e^{iS[x(t)]} \,, \tag{18}$$

where

$$X = e^{i\int dtJ(t)x(t)]} \,, \tag{19}$$

Therefore expand $e^{i\int dtJ(t)x(t)]}$ in a power series, which is then time-ordered.

$$\begin{aligned}\frac{z[J]}{z_0} &= 1 + i\int dtJ(t)\langle\, 0\,|\,\widehat{x}(t)\,|\,0\,\rangle + \cdots + \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int dt_1\cdots dt_n\, G_n(t_1,\ldots,t_n)J(t_1)\cdots J(t_n)\,,\end{aligned} \tag{20}$$

where the J's are just numbers that can be pulled out. Paradoxically, G_n is a symmetry function of $t_1,\ldots,t_n$.

2 Example: Harmonic Oscillation

$$S = \int_{-\infty}^{+\infty} dt \left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega_0^2 x^2\right) \quad (m = 1)\,. \tag{21}$$

$$H = \frac{p^2}{2m} + \tfrac{1}{2}\omega_0^2 x^2 = \frac{p^2}{2m}(1 - i\epsilon) + \tfrac{1}{2}\omega_0^2 x^2 (1 - i\epsilon)\,. \tag{22}$$

Do another Legendre transform back to a Lagrangian:

$$\begin{aligned} L_\epsilon &= \tfrac{1}{2}\dot{x}^2(1 + i\epsilon) - \tfrac{1}{2}\omega_0^2 x^2 (1 - i\epsilon) \quad \text{(next: do an 'integration by parts')} \\ &= -\tfrac{1}{2}x(\partial_t^2 + \omega_0^2 - i\epsilon\omega_0^2 + i\epsilon\partial_t^2)x + \cancelto{0}{\text{total derivative}} \\ &= -\tfrac{1}{2}x(\partial_t^2 + \omega_0^2 - i\epsilon)x \quad \text{(simplification } c\epsilon \sim \epsilon \text{ and collect terms)}\,. \end{aligned} \tag{23}$$

$$\begin{aligned} &S_\epsilon[x(t)] = -\tfrac{1}{2}\int dt dt' x(t) K(t,t') x(t') \quad \text{(which has a matrix structure)}\,, \\ &K(t,t') = \delta(t - t')(\partial_t^2 + \omega_0^2 - \epsilon)\,. \end{aligned} \tag{24}$$

Evaluate the path integral.

$$\begin{aligned} z_0 &= \int Dx \exp\left[-\frac{i}{2} x \cdot K \cdot x\right] \\ &= \frac{c}{\sqrt{\det K}} \end{aligned} \tag{25}$$

where c is some constant.

$$\begin{aligned} z[J] &= \int Dx \exp\left[-\tfrac{1}{2} x \cdot K \cdot x + iJ \cdot x\right] \\ &= \frac{c}{\sqrt{\det K}} \exp\left[\frac{i}{2} J \cdot K^{-1} \cdot J\right] \\ &= \frac{c}{\sqrt{\det K}} \exp\left[\frac{i}{2}\int dt dt' J(t) K^{-1}(t,t') J(t')\right]. \end{aligned} \tag{26}$$

where

$$\int dx_1 \cdots dx_n \exp\left[-\tfrac{1}{2}x_i A_{ij} x_j + J_i x_i\right] = \frac{(2\pi)^{1/2}}{\sqrt{\det K}} \exp\left[\tfrac{1}{2} J_i (A^{-1})_{ij} J_j\right]. \tag{27}$$

Now,

$$\int dt' K(t,t') K^{-1}(t,t') = \delta(t - t\text{''}) \tag{28}$$

or

$$K_{mn} K^{-1}_{nk} = \delta_{mk}\,. \tag{29}$$

Next, take the ratio:

$$\frac{z[J]}{z_0} = \exp\left[\frac{i}{2} J \cdot K^{-1} \cdot J\right] \tag{30}$$

$$= \exp\left[-\tfrac{1}{2}\int dt dt' J(t) G_F(t,t') J(t')\right]. \tag{31}$$

To evaluate,

a) Expand in powers of J.

b) Take derivatives.

Interpret K^{-1}.

Looking at (31), a one-point function, set $\frac{z[J]}{z_0} = 0$ since one derivative will leave one J, which is then to be set to zero.

Consider the two-point function defined by

$$\begin{aligned} G_F(t,t') &= G_2 \\ &= \frac{1}{z_0}\frac{1}{i^2}\frac{\delta^2 z[J]}{\delta J(t)\delta J(t')}\bigg|_{J=0} \\ &= -iK^{-1}(t,t')\,, \end{aligned} \tag{32}$$

the Feynman propagator for the harmonic oscillator. Then

$$K^{-1}(t,t') = iG_F(t,t')\,, \tag{33}$$

and

$$\frac{z[J]}{z_0} = \exp\left[-\tfrac{1}{2}J\cdot G_F\cdot J\right]. \tag{34}$$

From (24) and (28), K^{-1} satisfies

$$(\partial_t^2 + \omega_0^2 - i\epsilon)K^{-1} = \delta(t-t') \tag{35}$$

has solution (33)

On going to momentum space,

$$G_F(\omega) = \frac{i}{\omega^2 - \omega_0^2 + ik}\,. \tag{36}$$

Next, the n-point function. For general odd n, $G_0 = 0$. If the derivatives do not affect both of the J's in $z[J]/z_0$ then when $J \to 0$ the whole expression goes to zero.

For even n:

$$\begin{aligned} G_n &= \langle\, 0\,|\,T(\widehat{x}(t_1)\cdots\widehat{x}(t_n)\,|\,0\,\rangle \\ &= \sum \text{all poss. `contractions' between } \widehat{x}(t_i)\text{'s}\,, \end{aligned} \tag{37}$$

where a generic contraction pair is given as

$$\widehat{x}(t_i)\widehat{x}(t_j) = G_F(t_i,t_j)\,. \tag{38}$$

(The Wick Theorem proves this as a path-integral result for each pair of t_i, t_j and each pair is time-ordered.)

3 Example: 4-pt function

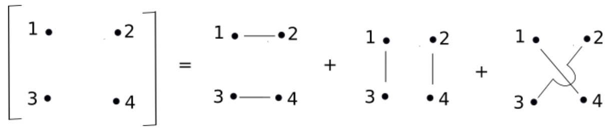

Figure 1. Example: 4-pt function.

$$= G_F(t_1,t_2)G_F(t_3,t_4) + G_F(t_1,t_3)G_F(t_2,t_4) + G_F(t_1,t_4)G_F(t_2,t_3)\,. \tag{39}$$

4 Time-ordered functions in field theory

3.3 Just replace the appropriate dynamics varaibles in QM by the appropriate dynamics varaibles in field theory.

$$G_n(x_1,\ldots,x_n) = \langle\,\Omega\,|\,T(\phi(x_1),\ldots,\phi(x_n))\,|\,\Omega\,\rangle\,, \tag{40}$$

where the x_i are spacetime points, and Ω represents the vacuum state.

Then, with $X = T(\phi(x_1),\ldots,\phi(x_n))$,

$$G_n(x_1,\ldots,x_n) = \frac{\int D\phi X\, e^{iS[\phi]}}{\int D\phi e^{iS[\phi]}}\,, \tag{41}$$

with boundary conditions

$$\phi(t,\mathbf{x}) \begin{cases} \longrightarrow 0, & t \to \pm\infty \\ \longrightarrow 0, & |\,\mathbf{x}\,| \to \infty \end{cases}\,, \tag{42}$$

for integration by parts and others to calculate end-point functions.

Next, the generating functional:

$$\begin{aligned} z[J] &= \int Dx \exp\left\{[iS[\phi] + i\int d^4x J(x)\phi(x)\right\}, \\ \frac{z[J]}{z_0} &= \left\langle\,\Omega\,|\,T\exp\left(i\int d^4x J(x)\phi(x)\right)|\,\Omega\,\right\rangle. \end{aligned} \tag{43}$$

So, we expand $\int d^4xJ(x)\phi(x))$ in a power series of ϕ and then apply time-ordering on them.

$$z_0 = z[J=0]\,. \tag{44}$$

5 Free Field Case

This case is similar to the harmonic oscillator case.

We begin with the Lagrangian

$$\mathscr{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2\,, \tag{45}$$

$$S = -\tfrac{1}{2}\int d^4x d^4x' \phi(x)K(x,x')\phi(x') \equiv -\tfrac{1}{2}\phi\cdot K\cdot\phi\,, \tag{46}$$

$$K = (-\partial^2 + m^2 - i\epsilon)\delta^{(4)}(x-x')\,, \tag{47}$$

$$z[J] = \int Dx \exp[-\frac{i}{2}\phi \cdot K^{-1} \cdot \phi + iJ \cdot \phi]$$
$$= \frac{c}{\sqrt{\det K}} \exp[\frac{i}{2} J \cdot K^{-1} \cdot J] , \quad (48)$$

where $z_0 = \sqrt{\det K}$ and

$$K^{-1}(x, x') = iG_F(x, x') , \quad (49)$$

and

$$\frac{z[J]}{z_0} = \exp[-\tfrac{1}{2} J \cdot G_F \cdot J] . \quad (50)$$

To calculate the n-pt functions.

By the Wick Theorem — everything goes through.

So, how to represent interactions?

6 Interacting theories

We start with an amended Lagrangian:

$$\mathscr{L} = \mathscr{L}_0 - \frac{\lambda}{4!}\phi^4 = \mathscr{L}_0 + \mathscr{L}_I , \quad (51)$$

where $\mathscr{L}_I$ has more generality. The Hamilton becomes:

$$H = H_0 + H_I , \quad (52)$$

where

$$H_I = -\int d^3x \mathscr{L}_I , \quad (53)$$

where $\partial \mathscr{L}_I / \partial \dot{\phi} \equiv 0$.

$$S[\phi] = S_0[\phi] + S_I[\phi] , \quad (54)$$

and

$$S_I[\phi] = \int d^4x \mathscr{L}_I = -\int dt H_I . \quad (55)$$

Now, to calculate the n-point function. The generating functional is

$$G_n = \frac{\int D\phi X \, e^{iS}}{\int D\phi e^{iS}} . \quad (56)$$

But because of the quadratic term in the integrand, we don't know how to do integral (56). However, we can treat this integral perturbatively, expanding it in a power series of λ.

$$G_n = \frac{\int D\phi X \, e^{iS}}{\int D\phi e^{iS}} = \frac{\int D\phi X \, e^{iS_0} e^{iS_I}}{\int D\phi e^{iS_0} e^{iS_I}} . \quad (57)$$

On expanding this last equation in a power series effectively reduces to the path integral for the free field, yielding

$$G_n = \frac{\langle 0 \,|\, T(X e^{iS_I}) \,|\, 0 \rangle}{\langle 0 \,|\, T(e^{iS_I}) \,|\, 0 \rangle} , \quad (58)$$

where the 0 state is the free-theory vacuum. To finish this, use Feynman diagrams.