Lecture 11: Computation of Correlation Functions in Perturbation Theory and Feynman Diagrams

P. Reany

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Abstract

This presentation is my read-along notes on the Lecture 11 from Hong Liu: MIT 8.323 Relativistic Quantum Field Theory I, Spring 2023. The fault for any inaccuracies in this presentation is most likely my own.

1 Some review

$$\langle\, \Omega \,|\, T(X) \,|\, \Omega \,\rangle = \frac{\int D\phi X\, e^{iS}}{\int D\phi e^{iS}}\,, \tag{1}$$

which applies to any theory and to any X.

2 Interacting Theories

For interacting theories

$$S = S_0 + S_I\,, \tag{2}$$

$$S_I = \int d^4x \mathscr{L}_I = -\int dt \mathscr{H}_I\,, \tag{3}$$

with time-varying restriction. Treat S_I as small and then expand.

$$\langle\, \Omega \,|\, T(X) \,|\, \Omega \,\rangle = \frac{\int D\phi X\, e^{iS_0} e^{iS_I}}{\int D\phi e^{iS_0} e^{iS_I}} = \frac{\langle\, 0 \,|\, T(X e^{iS_I}) \,|\, 0 \,\rangle_0}{\langle\, 0 \,|\, T(e^{iS_I}) \,|\, 0 \,\rangle_0}\,, \tag{4}$$

where the 0 represents the vacuum state in free theory. This equation is (4.31) in Preskin and Schroeder, and represents the exact expression, which write the correlation function of an interacting theory in terms of the free theory.

However, we can treat this integral (4) perturbatively, expanding it in a power series of S_I:

$$\begin{aligned} G_n &= \langle\, 0 \,|\, T(X) \,|\, 0 \,\rangle - i \int dt \langle\, 0 \,|\, T(X H_I) \,|\, 0 \,\rangle + \dots \\ &= 1 - i \int dt \langle\, 0 \,|\, T(X H_I) \,|\, 0 \,\rangle + \dots\,, \end{aligned} \tag{5}$$

where the first term is the zeroth-order term, and the second term is the first-order term. Continuing,

$$G_n = G_n^{(0)} + i \langle\, 0 \,|\, T X \,|\, 0 \,\rangle \int dt \langle\, 0 \,|\, T H_I \,|\, 0 \,\rangle - i \int dt \langle\, 0 \,|\, T(X H_I) \,|\, 0 \,\rangle + \dots\,, \tag{6}$$

which can be evaluated using the Wick Theorem. To simplify this calculation, we introduce the Feynman diagram.

Let's try the case $\mathscr{L}_I = -\frac{\lambda}{4!}\phi^4$. Consider a 2-point function (Feynman function).

$$G_2(x_1{}^{\text{‘}} - x_2) = \langle\, \Omega \,|\, T\phi(x_1)\phi(x_2) \,|\, \Omega \,\rangle \,. \tag{7}$$

In the 0-th order H_I integrates over 3-D space and time.

$$G_2 = G_F^{(0)}(x_1{}^{\text{‘}} - x_2) + i\frac{\lambda}{4!}G_F^{(0)}(x_1{}^{\text{‘}} - x_2)\int d^4x \langle\, 0 \,|\, T\phi^4(x) \,|\, 0 \,\rangle - \frac{i\lambda}{4!}\int d^4 \langle\, 0 \,|\, \phi(x_1)\phi(x_2)\phi^4(x) \,|\, 0 \,\rangle + O(\lambda^2)\,. \tag{8}$$

Remark: All terms in the expansion can be evaluated by the repeated application of Wick's Theorem to 2-point functions. The final result has the form of the sum over the product of $G_F^{(0)}$'s plus integration.

$$\begin{aligned} G_F(x_1 - x_2) = G_F^{(0)} &+ \frac{i\lambda}{4!}G_F^{(0)}(x_{12})\,3\int d^4x\, G_F^{(0)}(0)G_F^{(0)}(x-x) - \frac{i\lambda}{4}G_F^{(0)}(x_{12})\,3\int d^4x\,(G_F^{(0)}(0))^2 \\ &- \frac{i\lambda}{4!}4\cdot 3\int d^4x\, G_F^{(0)}(x_1 - x)G_F^{(0)}(x_2 - x)G_F^{(0)}(0) + O(\lambda^2)\,. \end{aligned} \tag{9}$$

The second and third terms cancel.

$$\overline{x \qquad\qquad\qquad\qquad y} \quad \longleftrightarrow \quad G_F^{(0)}(x-y) \tag{10}$$

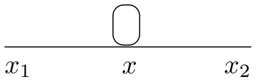
The diagrammatic form is:

the first term is: $\overline{x_1 \qquad\qquad\qquad\qquad x_2}$

the 2nd term is: $\frac{i\lambda}{8}\ \overline{x_1 \qquad\qquad\qquad\qquad x_2} \quad 8x$

the 3rd term is: $-\frac{i\lambda}{8}\ \overline{x_1 \qquad\qquad\qquad\qquad x_2} \quad 8x$

the 4th term is: $-\frac{i\lambda}{2}$

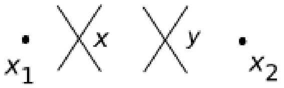


+ O(λ^2) .

In practice, it's best to first draw the Feynman diagrams and then write down the formal computations from them.

Example of order λ^2:

$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \int d^4x\, d^4y \langle\, 0 \,|\, \phi(x_1)\phi(x_2)\phi^4(x))\phi^4(y) \,|\, 0 \,\rangle \,. \tag{11}$$



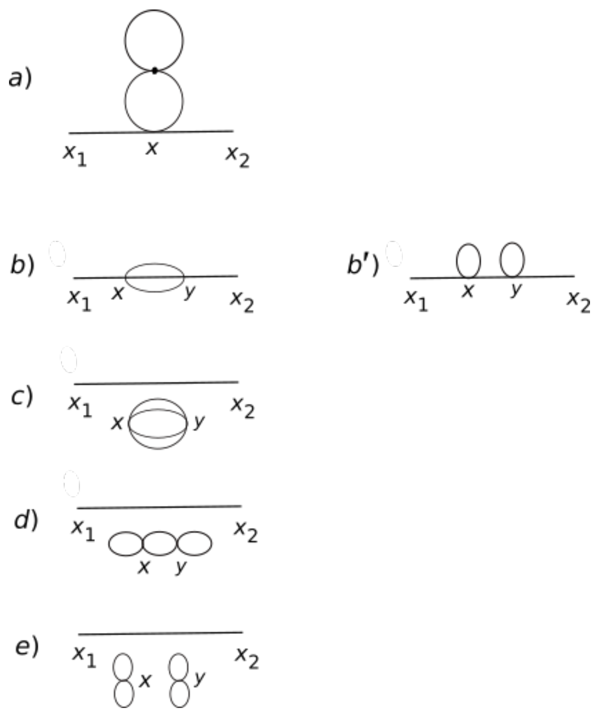


Figure 1. Modes of interactions.

$$\text{a) } -\tfrac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2 \cdot 2\cdot 4\cdot 3\cdot 4\cdot 3 = -\lambda^2\left(\frac{1}{2}\right)^2$$

$$\text{b) } -\tfrac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2 \cdot 2\cdot 4\cdot 4\cdot 3\cdot 2 = -\lambda^2\left(\frac{1}{3!}\right)^2$$

b') ...

$$\text{c) } \tfrac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2 \cdot 4\cdot 3\cdot 2 = -\frac{\lambda^2}{2}\frac{1}{4!}$$

$$\text{d) } -\tfrac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2 2\cdot 3\cdot 4\cdot 3 = -\frac{\lambda^2}{16}$$

$$\text{e) } -\tfrac{1}{2}\frac{\lambda^2}{(4!)^2}\cdot 3^2 = -\frac{\lambda^2}{2}\left(\frac{1}{8}\right)^2$$

A better way to count the combinations

(1) $\lambda^n : \dfrac{1}{n!}$ from expanding the exponential, n vertices. On permuting the n interacting vertices given $n!$. They cancel, modulo symmetries permuting vertices.

(2) For each vertex $\dfrac{-i\lambda}{4!}$: if each ϕ in ϕ^4 contracts differently. Permuting 4ϕ's implies 4!. Modulo the symmetries in permuting ϕ's coming from the same vertex.

So, we can permute both vertices and ϕ's. The $n!$ factorials will cancel each vertex as $(-i\lambda\ )$. Then divide by the symmetry factors from permuting vertices and the legs.

Return to (a) dagram: There's no symmetry between x and y. But a function of two legs from x and a function of two legs from y is symmetric. So we divide by $\left(\frac{1}{2}\right)^2$.

Rules for writing the expressions from the diagrams (Feynman Rules):

1. For each external point assocate a line segment:

the first term is: $\overline{x_1 \qquad\qquad\qquad\qquad} \quad 1$

2. For the propagator:

$$\overline{x \qquad\qquad\qquad\qquad y} \quad \longleftrightarrow \quad G_F^{(0)}(x-y)$$

3. For each vertex:

$$\mathsf{X}\!\leftarrow x \qquad -i\int d^4x$$

4. Divide by the symmetry factors.

(b) Example:

$$\frac{(-i\lambda)^2}{3!}\int d^4x\, d^4y\, [G_F^{(0)}(x-y)]^3$$

3 Convert to momentum space

We need the Fourier transform for an n-point function.

$$\int d^4x_1\, d^4x_n\, e^{-p_1\cdot x_1\cdots - ip_n\cdot x_n}\, G_n(x_1\cdots x_n)$$
$$= (2\pi)^4\delta^{(4)}(p_1+\cdots+p_n)G_n(p_1\,\ldots\,,p_n)\,. \tag{12}$$

From translation symmetry:

$$G_n(x_1\cdots, x_n) = G_n(x_1-x_n, x_2-x_n, \ldots, 0)\,, \tag{13}$$

where we have chosen x_n as the reference point. So, when we substitute this thing into the LHS of (12), we get a delta function.

$$G_F^{(0)}(k) = \frac{-i}{k^2+m^2-i\epsilon} \qquad \xrightarrow{\qquad\qquad k \qquad\qquad} \tag{14}$$

At each vertex

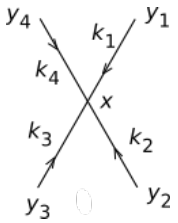


Figure 2. Momentum interactions. x is contracted with all four y's.

$$\int d^4x\, G_F^{(0)}(x-y_1)\, G_F^{(0)}(x-y_2)G_F^{(0)}(x-y_3)\, G_F^{(0)}(x-y_4) \tag{15}$$

plug in the expression in momentum space.

$$\int d^4k_1\, d^4 e^{ik_1(x-y_1)}\, G_F^{(0)}(k_1) \tag{16}$$

yields a delta function

$$(2\pi)^4 k_1 \delta^{(4)}(k_1 +_2 + k_3 + k_4)\,. \tag{17}$$

So, momentum is conserved at each vertex.

Momentum space Feynman Rules:

To compute $G_n(p_1, \ldots, p_n)$

1) at each external point $\xleftarrow{\qquad p_i \qquad}$ 1.

2) for each propagator

$$\xrightarrow{\qquad p \qquad} \quad \frac{-i}{k^2 + m^2 - i\epsilon} \tag{18}$$

3) for each vertex

$$\mathsf{X} \leftarrow -i\lambda$$

and impose momentum conservation at each vertex.

4) Integrate over each undertermined momentum.

$$\int \frac{d^4p_i}{(2\pi)^4}\,. \tag{19}$$

5) Divide by symmetry factor.

Example: Redo (b) but in momentum space.

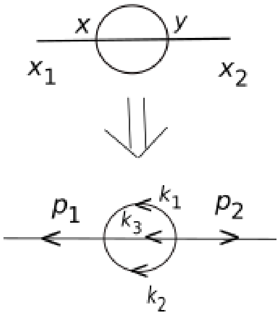


Figure 3. Example momentum interactions. From position space to momentum space.

Now, impose momentum conservation.

$$p_1 = k_1 + k_2 + k_3\,, \tag{20}$$
$$p_2 = -(k_1 + k_2 + k_3) = -p_1\,. \tag{21}$$

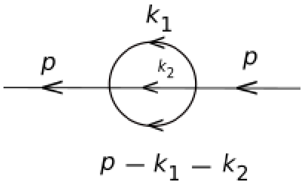


Figure 4. Example momentum flow.

$$\frac{(-i\lambda)^2}{3!}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{-i}{k_1^2+m^2-i\epsilon}\frac{-i}{k_2^2+m^2-i\epsilon}\frac{-i}{(p-k_1-k_2)^2+m^2-i\epsilon}\left(\frac{-i}{p^2+m^2-i\epsilon}\right)^2 . \tag{22}$$