Lecture 8: Path Integrals

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Abstract

This presentation is my read-along notes on the Lecture 8 from Hong Liu: MIT 8.323 Relativistic Quantum Field Theory I, Spring 2023 The fault for any inaccuracies in this presentation is strictly my own.

1 Some review

The path-integral approach to Action.

Previously:

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \,, \tag{1}$$

for a particle with interaction term $-\dfrac{\lambda}{4!}\phi^4$. The treatment for this theory will act as the pattern for all subsequent versions of interacting Lagrangians.

In the Schrödinger picture, we have for the

S-matrix:

$$S_{\beta\alpha} = \langle\, \beta \mid U(+\infty, -\infty) | \alpha \,\rangle \,, \tag{2}$$

where α is the initial state and β is the final state and where $U(+\infty, -\infty)$ is the *evolution operator*. In the Heisenberg picture we have

$$\begin{aligned} S_{\beta\alpha} &= \langle\, \beta, +\infty \mid \alpha, -\infty \,\rangle \,, \\ S_{\beta\alpha} &= \delta_{\beta\alpha} + iT_{\beta\alpha} \,, \end{aligned} \tag{3}$$

where

$$iT_{\beta\alpha} = i(2\pi)^4 \delta^{(4)}(p_\alpha - p_\beta) M_{\alpha\to\beta} \,, \tag{4}$$

where p_α is the total initial momentum and p_β is the total final momentum; and where $M_{\alpha\to\beta}$ is the physical part of the scattering amplitude.

In LSZ, $M_{\alpha\to\beta}$ can be obtained from the time-ordered function

$$G_n \equiv \langle\, \Omega \mid T\phi(x_1)\cdots\phi(x_n) \,|\, \Omega \,\rangle \,, \tag{5}$$

where $\phi(x_1)\cdots\phi(x_n)$ are listed in reverse time order, and Ω is the vacuum state, and where k is the total number of particles. Also, Ω represents the vacuum. The operator T means time-ordering, which place the largest time ealiest.

Computation of the correlation function can be performed with perturbation theory.

The 'Equation of motion' is:

$$(\partial^2 - m^2)\phi = \frac{\lambda}{6}\phi^3\,, \tag{6}$$

which has no exact solution. This is an interaction term ϕ^3. To solve (6) approximately, treat λ as a small parameter. For a perturbative solution, we make the substitution,

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots\,, \tag{7}$$

which is then substituted into the EOM, and then the terms of same power in λ are set equal.

$$(\partial^2 - m^2)\phi_0 = 0 \quad \text{for the free field}\,, \tag{8}$$

Then the first-order equation is

$$(\partial^2 - m^2)\phi_1 = \frac{1}{6}\phi_0^3\,, \tag{9}$$

etc. Then we solve this set of equations, first to last, to get ϕ.

Sinilarly, we can solve

$$|\,\Omega\,\rangle = |\,0\,\rangle + \lambda\,|\,\Omega_1\,\rangle + \lambda^2\,|\,\Omega_2\,\rangle + \cdots\,. \tag{10}$$

So, G_n can be reduced to a calculation in free theory, since, ultimately, everything erduces to a function of ϕ_0. But this method isn't easy. However, we have better techniques.

Two approaches:

(1) Use the *interaction picture.*

$$H = H_0 + H_I\,, \tag{11}$$

$|\,\psi_I(t)\,\rangle$, $O_I(t)$ which is the state and an operator, respectively, of the interaction picture.

From the Schrödinger picture

$$|\,\psi_I(t)\,\rangle = e^{iH_0t}\,|\,\psi_s(t)\,\rangle \tag{12}$$

$$O_I(t) = e^{iH_0t}O_S e^{-iH_0t}\,, \tag{13}$$

which is the Heisenberg picture (or "operators"). See Peskin and Schroeder [*An Introduction to Quantum Field Theory*], Sec. 4.2.

2 The Path Integral

(2) A more straightforward alternative is the *path integral* formulation of quantum mechanics.

Example: For non-relativistic QM:

$$H = \frac{P^2}{2m} + V(x)\,. \tag{14}$$

1) In contrast, for the path-integral version, we start with the propagator:

$$K(x,t;x',t') = \langle\, x,t \mid x',t'\,\rangle = \left\langle\, x \mid e^{-iH(t-t')}|x'\,\right\rangle\,, \tag{15}$$

(the latter being in the Schrödinger picture), which we solve to get $\psi(x,t)$, thusly

$$\psi(t,x) = \int dx'\, K(x,t;x',t')\psi(t',x')\,. \tag{16}$$

So, how to compute K?

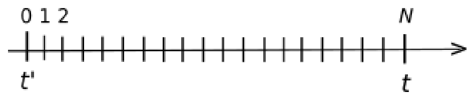

Figure 1.

$$t_0 = t', \quad t_N = t, \quad \Delta t = \frac{t - t'}{N} \, . \tag{17}$$

And we take the limit as $N \to \infty$ and thus $\Delta t \to 0$.

Note:

$$e^{-iH(t-t')} = \left(e^{-iH\Delta t}\right)^N \, , \tag{18}$$

Then, into

$$\left\langle \, x \mid e^{-iH\Delta t} e^{-iH\Delta t} \cdots e^{-iH\Delta t} | x' \, \right\rangle \, , \tag{19}$$

insert between each factor the integral: $\int dx_i \, | \, x_i \, \rangle \, \langle \, x_i \, | = 1$, so that,

$$K = \int dx_1 \cdots dx_{N-1} \left\langle \, x \mid e^{-iH\Delta t} | x_{N-1} \, \right\rangle \cdots \left\langle \, x_1 \mid e^{-iH\Delta t} | x' \, \right\rangle \, . \tag{20}$$

Each factor can be calculated

$$e^{-iH\Delta t} = e^{-iH\Delta t \left(\frac{\widehat{p}^2}{2m} + V(\widehat{x}) \right)} \approx e^{-iH\Delta t \frac{\widehat{p}^2}{2m}} e^{-iH\Delta t V(\widehat{x})} (1 + \mathrm{O}(\Delta t^2)) \, , \tag{21}$$

then add all the contributions together.

Based on the Baker-Campbell-Hausdorff (BCH) formula:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B] + \cdots} \, , \tag{22}$$

$$\left\langle x_{i+1} \middle| e^{iH\Delta t \frac{\widehat{p}^2}{2m}} e^{-iH\Delta t V(\widehat{x})} \middle| x_i \right\rangle = \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{im}{2} \left(\frac{x_{i+1} - x_i}{\Delta t} \right)^2 \Delta t - i\Delta t V(x_1) \right] \tag{23}$$

Consider $x(t_i) = x_i$, then,

$$\left\langle x_{i+1} \middle| e^{iH\Delta t \frac{\widehat{p}^2}{2m}} e^{-iH\Delta t V(\widehat{x})} \middle| x_i \right\rangle = \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[i\Delta t L(x, \dot{x}) \Big|_{t_i} \right] , \tag{24}$$

where L is the Lagrangian. So,

$$\begin{aligned} K &= \lim_{N \to \infty} \left(\frac{m}{2\pi i \Delta t} \right)^{N/2} \int dx_1 \cdots dx_{N-1} \exp\left[i\Delta t \sum_{i=0}^{N-1} L(x, x') \Big|_{t_i} \right] \\ &= \int_{x(t')=x'}^{x(t)=x} Dx(t) \exp\left[i \int_{t'}^{t} dt'' L(x, x') \right] , \end{aligned} \tag{25}$$

where $Dx(t)$ integrates over all possible functions $x(t)$ satisfying $x(t') = x'$ and $x(t) = x$.

Physical Interpretation.

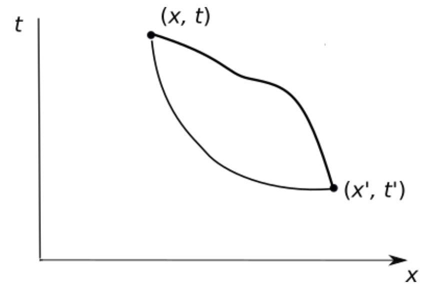

Figure 2. We want all paths between (x, t) and (x', t'), weighted by e^{iS}.

We want all paths between (x, t) and (x', t'), weighted by e^{iS}, where

$$S = \int_{t'}^{t} dt\, L(x, \dot{x}) \tag{26}$$

is the action. Then, an alternative (Hamiltonian) form for K is

$$K = \int_{x(t')=x'}^{x(t)=x} Dx(t)\, Dp(x) \exp\left[i \int_{t'}^{t} dt''(p\dot{x} - H)\right]. \tag{27}$$

Example: The simple case of a free particle with $s = \frac{1}{2}\dot{q}^2$

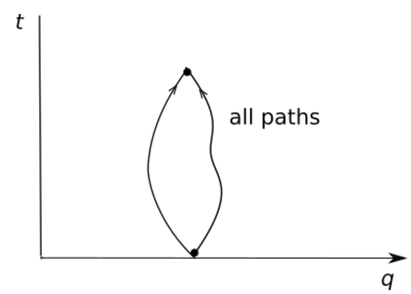

Figure 3. Case of a free particle.

Here, Z_T is the amplitude we wish to calculate, where the particle returns to its initial point, and T is the final time.

$$\begin{aligned} Z_T &= \langle\, 0, T \mid 0, 0\,\rangle \\ &= \lim_{N\to\infty} \ldots\ldots \qquad \text{(Homework!)} \\ &= \left(\frac{m}{2\pi i T}\right)^{1/2}. \end{aligned} \tag{28}$$

But there is a simpler method.

$$\begin{aligned} S &= \int_0^T dt \tfrac{1}{2} m \dot{q}^2 \qquad \text{(Integrate by parts.)} \\ &= \frac{m}{2} \int_0^T dt q(-\partial_t^2) q\,. \end{aligned} \tag{29}$$

Boundary term to zero since $q_i = q_f = 0$.

$$S = \tfrac{1}{2}\int_0^T dtdt' q(t)(-m\delta(t-t')\partial_{t'}^2)q(t')\,. \tag{30}$$

This is an identity transformation (by virtual emplacement).

$$S = \tfrac{1}{2}\int_0^T dtdt' q(t)K(t,t')q(t')\,, \tag{31}$$

where $K(t,t') \equiv -m\delta(t-t')\partial_{t'}^2$. Now, treat t, t' as continuous indexes. Then, in the extpression

$$q(t)K(t,t')q(t')\,, \tag{32}$$

$K(t,t')$ is a matrix and $q(t)$, $q(t')$ are vectors. So, in simplified notation:

$$Z_T = \int Dq(t)\exp\left(\frac{i}{2}\int dtdt' q\cdot K\cdot q\right)\,, \tag{33}$$

which is a gaussian integral.

Recall that

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{ax^2}{2}\right) = \sqrt{\frac{2\pi}{a}}\,, \tag{34}$$

and

$$\int_{-\infty}^{\infty} dx_1\cdots dx_n \exp\left(-\tfrac{1}{2}x_m A_{mn} x_n\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\,, \tag{35}$$

Hence,

$$Z_T = \frac{C}{\sqrt{\det K}}\,, \tag{36}$$

where C is a constant, and $\det K$ is the determinant of K in the space of (uncountable) infinite-dimensional vectors [in the space of functions]. Now, C can be divergent. Not all divergences are important in QFT.

So, how do we calculate $\det K$? Note,

$$\det A = \prod_i \lambda_i\,, \tag{37}$$

where λ_i is an eigenvalue of K.

$$A_{mn}x_n = \lambda x_m\,. \tag{38}$$

So,

$$\int_0^T dt' K(t,t') f_i(t') = \lambda_i f(t)\,, \tag{39}$$

where, from (38) $n \to t'$ and the sum over n is replaced by the integral. Then,

$$f_i(0) = f_i(T) = 0\,. \tag{40}$$

Therefore,

$$f_j(0) = \sin\frac{\pi j t}{T} = 0\,, \tag{41}$$

$$\lambda_j = \frac{mj^2\pi^2}{T^2}\,, \quad j = 1, 2, 3, \dots\,, \tag{42}$$

which implies that

$$\det K = \prod_{j=1}^{\infty} m\frac{j^2\pi^2}{T^2}\,. \tag{43}$$

3 Remarks

(1) This new formulation of QM is equivalent to the Schrödinger formulation. The K satisfies the Schrödinger equation

$$i\partial_t K = -\frac{1}{2m}\partial_x^2 K + V(x)K\,. \tag{44}$$

(2) Contrast classical mechanics, in which the classical path is realized:

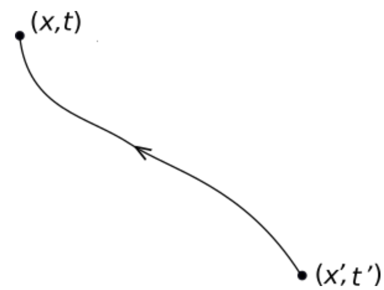

Figure 4. Classical path.

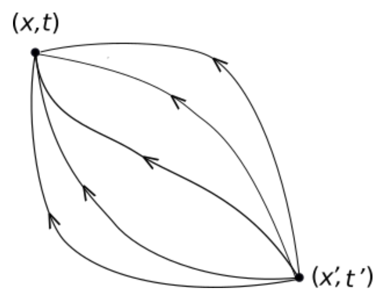

Figure 5. Many paths sum.