Lecture 9: Path Integrals for QFT

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June 22, 2024

Abstract

This presentation is my read-along notes on the Lecture 9 from Hong Liu: MIT 8.323 Relativistic Quantum Field Theory I, Spring 2023 The fault for any inaccuracies in this presentation is strictly my own.

1 Some review

The path-integral formulation to QM. Classically, we envision the particle moving in a straight line from point (x', t') to (x, t) . In QM, we visualize the situation as a sum over paths.

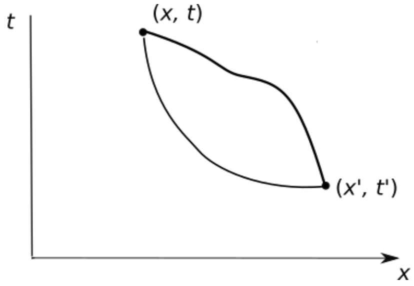

Figure 1. We want all paths between (x', t') and (x, t), weighted by e^{iS}.

$$\langle\, x, t \mid x', t' \,\rangle = \text{sum over all paths}\,, \tag{1}$$

with weights

$$\begin{aligned}\langle\, x, t \mid x', t' \,\rangle &= \sum_{\text{paths}} e^{\frac{i}{\hbar} S[\text{path}]} \\ &\to \int_{x(t')=x'}^{x(t)=x} Dx(t)\, e^{\frac{i}{\hbar} S[x(t)]}\,,\end{aligned} \tag{2}$$

which, for a 1-dimensional particle is,

$$S[x(t)] = \int_{t'}^{t} dt'' (\tfrac{1}{2} m \dot{x}^2 - V(x))\,. \tag{3}$$

Also

$$\int Dx(t) \equiv \lim_{N\to\infty} \Big(\frac{m}{2\pi i \Delta t} \Big)^{N/2} \int dx_1 \cdots dx_{N-1}\,. \tag{4}$$

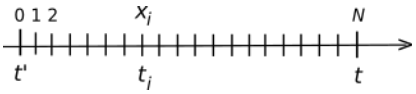

Figure 2.

Free-particle example:

$$S = \int \tfrac{1}{2} m\dot{x}^2 \, dt \,. \tag{5}$$

Gaussian integral[1]

$$\int Dx(t) \exp\big[\frac{i}{2}\int dt dt' x(t) K(t,t') x(t')\big] = \frac{c}{\sqrt{\det(K)}} \,. \tag{6}$$

Generalization to field theory. For this

$$\widehat{x}(t) \to \widehat{\phi}(t, \mathbf{x}) \,, \tag{7}$$

where $\widehat{\phi}$ is the field variable.

$$\begin{aligned} \langle\, x, t \mid x', t' \,\rangle &\to \langle\, \phi(\mathbf{x}), t \mid \phi'(\mathbf{x}), t' \,\rangle \\ &= \Big\langle\, \phi(\mathbf{x}) \mid e^{-iH(t-t')} \,|\, \phi'(\mathbf{x}) \,\Big\rangle \,, \end{aligned} \tag{8}$$

where we view $\mathbf{x}$ as merely a label. Split the time interval $t - t'$ into many time increments.

$$H = \int d^3x \mathcal{H} \,, \tag{9}$$

where

$$\mathcal{H} = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) \,. \tag{10}$$

Then we have that

$$\Big\langle\, \phi(\mathbf{x}) \mid e^{-iH(t-t')} \,|\, \phi'(\mathbf{x}) \,\Big\rangle = \int_{\phi(t',\mathbf{x})=\phi'(\mathbf{x})}^{\phi(t,\mathbf{x})=\phi(\mathbf{x})} D\phi(\mathbf{x}, t) e^{iS[\phi]} \,, \tag{11}$$

and QFT reduces to calculating this last path integral. And action

$$S[\phi] = \int d^4x \, \mathcal{L}(\phi, \dot{\phi}) \,, \tag{12}$$

where

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi \, \partial^\mu\phi - V(\phi) \tag{13}$$

for a scalar field.

$$Dx(t) = \prod_i dx_i = \prod_t dx_t \quad \text{going from discreet to continuous} \,. \tag{14}$$

$$\begin{aligned} D\phi(\mathbf{x}, t) &= \prod_{\mathbf{x}} D\phi_{\mathbf{x}}(t) \quad \text{where } \mathbf{x} \text{ is just a label} \\ &= \prod_{t,\mathbf{x}} d\phi_{\mathbf{x},t} \,. \end{aligned} \tag{15}$$

Other than the additional “label” $\mathbf{x}$, the path integral for then scalar field is essentially identical to that in QM. For the free theory, $V(\phi) = \frac{1}{2}m^2\phi^2$. The path integral is again a Gaussian.

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[1] The integration follows the example of $\int_{-\infty}^{\infty} dx \, e^{ix^2}$.

2 A trick to the rescue!

Our goal is to compute the vacuum expectation value of the time-ordered correlation function:

$$\langle\, 0 \,|\, T(\phi(x_1) \cdots \phi(x_n) \,|\, 0 \,\rangle\,. \tag{16}$$

[I'm not sure what the professor placed in the bra and ket, because they look like Ωs, but since he said 'vacuum' I used zeros.]

Let's do it in QM first, and then generalize it to QFT. Warning! There's a trick to be used ahead.

The analog of (16) is

$$G_n = \langle\, 0 \,|\, T(\widehat{x}(t_1) \cdots \widehat{x}(t_n)) \,|\, 0 \,\rangle\,. \tag{17}$$

How to do time-ordering for the path integral?

First, consider $t_1 \in (t', t)$.

$$\begin{aligned} \langle\, x, t \,|\, \widehat{x}(t_1) \,|\, x', t' \,\rangle &= \int dx_1 \,\langle\, x, t \mid x_1, t_1 \,\rangle \,\langle\, x_1, t_1 \,|\, \widehat{x}(t_1) \,|\, x', t' \,\rangle \\ &= \int dx_1 x_1 \,\langle\, x, t \mid x_1, t \,\rangle \,\langle\, x_1, t_1 \,|\, x', t' \,\rangle\,. \end{aligned} \tag{18}$$

where x_1 is an eigenvalue that is pulled to the left.

Most of the tricks of QM reduce to one trick! Insert identities.

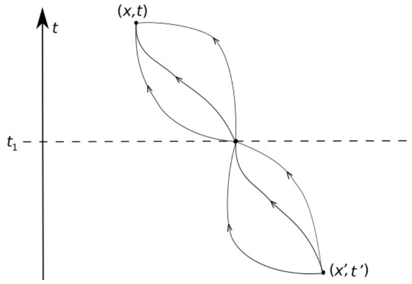

Figure 3. We consider the simplest case of just one intermediate point.

From (18) we continue:

$$= \int_{(x',t')}^{(x,t)} Dx(t)\, x(t_1)\, e^{iS[x(t)]}\,. \tag{19}$$

Suppose now two operators.

$$\begin{aligned} &\langle\, x, t \,|\, T(\widehat{x}(t_1)\widehat{x}(t_2)) \,|\, x', t' \,\rangle \qquad (t_1 > t_2) \\ &\qquad = \int_{(x',t')}^{(x,t)} Dx(t) x(t_1) x(t_2)\, e^{iS[x(t_1)]}\,. \end{aligned} \tag{20}$$

Time ordering naturally arises in path integrals. For any $t_1, \ldots, t_n \in (t', t)$

$$\langle\, x, t \,|\, T(\widehat{x}(t_1) \cdots \widehat{x}(t_n)) \,|\, x', t' \,\rangle = \int_{(x',t')}^{(x,t)} Dx(t) x(t_1) \cdots x(t_n)\, e^{iS[x(t)]}\,, \tag{21}$$

which is a key formula. So, how to do this for the vacuum.

3 Vacuum Correlation Functions

G_n for arbitrary $t_1 \cdots t_n \in (-\infty, +\infty)$ and for $X \equiv T(\widehat{x}(t_1) \cdots \widehat{x}(t_n))$:

$$G_n = \langle\, 0 \,|\, X \,|\, 0 \,\rangle \,. \tag{22}$$

The trick here is to insert the indentity (a complete set of states) into this — in the right place. We insert

$$1 = \int dx \,|\, x, t \,\rangle \,\langle\, x, t \,| = \int dx' \,|\, x', t' \,\rangle \,\langle\, x', t' \,| \;, \tag{23}$$

with $t \to +\infty$, $t' \to -\infty$.

$$\begin{aligned} G_n &= \lim_{\substack{t \,\to\, +\infty \\ t' \to -\infty}} \int dx\, dx' \langle\, 0 \,|\, x, t \,\rangle \,\langle\, x, t \mid X \,|\, x', t' \,\rangle \,\langle\, x', t' \mid 0 \,\rangle \\ &= \int dx\, dx' \psi_0^*(x) \psi_0(x') \int_{(x',t')}^{(x,t)} Dx(t) x(t_1) \cdots x(t_n)\, e^{iS[x(t)]} \;, \end{aligned} \tag{24}$$

where $\langle\, x', t' \mid 0 \,\rangle = \psi_0(x')$, the ground-state wavefunction. And we have made the assumption that the ground state is time independent. But this trick applies ot any state, not just to the vacuum state.

For the vacuum there's another trick to make two additional integrals unnecessary. Why? For abitrary wavefunctions, the formal result (24) may not be solvable. So, at this point we forget about $G_n = \langle\, 0 \,|\, X \,|\, 0 \,\rangle$.

4 The Unity Operator

Consider the unity operator expressed in terms of a complete set of energy eigenstates:

$$\mathbb{1} \equiv \sum_m |\, m \,\rangle \,\langle\, m \,| \;. \tag{25}$$

We can insert this unity operator on both sides of the X in the expression

$$\lim_{\substack{t \,\to\, +\infty \\ t' \to -\infty}} \langle\, x, t \,| X \,| x', t' \,\rangle = \lim_{\substack{t \,\to\, +\infty \\ t' \to -\infty}} \sum_{n,m} \langle\, x, t \mid m \,\rangle \,\langle\, m \,| X \,|\, n \,\rangle \,\langle\, n \mid x', t' \,\rangle \;. \tag{26}$$

We are interested in the $n = m = 0$ values. Going from the Heisenberg picture on the left to the Schrödinger picture on the right, we have for the simplified situation for the rightmost factor of (26)

$$\begin{aligned} \lim_{t' \to -\infty} \langle\, n \mid x', t' \,\rangle &= \lim_{t' \to -\infty} \left\langle\, n \mid e^{iHt'} \,|\, x' \,\right\rangle \\ &= \lim_{t' \to -\infty} e^{iE_n t'} \,\langle\, n \mid x' \,\rangle \;, \end{aligned} \tag{27}$$

where we have pulled out the energy eigenvalue factor.

Now, imagine that E_n has a small imaginary part

$$E_n \longrightarrow E_n(1 - i\epsilon) \tag{28}$$

or $H \longrightarrow H(1 - i\epsilon)$. For $E_n > E_0$ and $n > 0$

$$\lim_{t' \to -\infty} e^{iE_n t'} \,\langle\, n \mid x' \,\rangle = \lim_{t' \to -\infty} e^{iE_n(1-i\epsilon)t'} \,\langle\, n \mid x' \,\rangle \;. \tag{29}$$

Therefore for any state $n > 0$, $e^{\epsilon E_n t'} \to 0$.

Thus,

$$\lim_{t' \to -\infty} e^{iE_n(1-i\epsilon)t'} \langle\, n \mid x' \,\rangle = \begin{cases} e^{iE_0(1-i\epsilon)t'} \langle\, 0 \mid x' \,\rangle \\ 0 \qquad n \neq 0, \quad (t' \to -\infty) \end{cases} . \tag{30}$$

Also,

$$\lim_{t \to +\infty} \langle\, x, t \mid m \,\rangle = \lim_{t \to +\infty} e^{-iE_n(1-i\epsilon)t} \langle\, x \mid m \,\rangle = \begin{cases} e^{-iE_0(1-i\epsilon)t} \langle\, x \mid 0 \,\rangle\,, \quad m = 0\,, \\ 0 \qquad m \neq 0 \end{cases} , \tag{31}$$

where $\langle\, 0 \mid x' \,\rangle$ and $\langle\, x \mid 0 \,\rangle$ are ground-state wavefunctions.

$$\lim_{\substack{t \to +\infty \\ t' \to -\infty}} \langle\, x, t \,| X \,| x', t' \,\rangle = \psi_0^*(x')\psi_0(x) G_n e^{-iE_0(1-i\epsilon)(t-t')} \,. \tag{32}$$

where $G_n = \langle\, 0 \,| X \,| \, 0 \,\rangle$ and X is arbitrary, so we can set it to unity.

Or,

$$\lim_{\substack{t \to +\infty \\ t' \to -\infty}} \langle\, x, t \,|\, x', t' \,\rangle = \psi_0(x)\psi_0^*(x') 1 e^{-iE_0(1-i\epsilon)(t-t')} \,, \tag{33}$$

where $1 = \langle 0 \mid 0 \rangle$.

5 The Correlation Function

Thus we arrive at the beautiful Correlation Function formula:

$$G_n = \lim_{\substack{t \to +\infty \\ t' \to -\infty}} \frac{\langle\, x, t \,|\, X \,| x', t' \,\rangle}{\langle\, x, t \,| x', t' \,\rangle} \,, \tag{34}$$

where the values of x, x' will cancel in the ratios. To calculate G_n, we can take the limit as $x, x' \to 0$ and the limit as $\epsilon \to 0$ after the integration.

Just one more trick, sir!

6 Generating Function (even more cleverness)

Let

$$z_n = \int dx\, e^{i\lambda f(x)} x^n \,. \tag{35}$$

Consider

$$z(a) \equiv \int dx\, e^{i\lambda f(x) + iax} \,. \tag{36}$$

Then

$$z(a) = \sum_{n=0}^{\infty} \frac{(ia)^n}{n!} z_n \qquad \text{a generating function} \,. \tag{37}$$

Therefore,

$$z_n = \frac{1}{i^n} \frac{\partial^n z(a)}{\partial a^n}\bigg|_{a=0} \,. \tag{38}$$

$$G_n = \langle\, 0 \,|\, T(\widehat{x}(t_1) \cdots \widehat{x}(t_n)) \,|\, 0 \,\rangle \,. \tag{39}$$

Consider,

$$z[J(t)] = \int Dx(t)\, e^{iS[x(t)] + i\int_{-\infty}^{\infty} dt J(t) X(t)} \,. \tag{40}$$