

Galilean Covariance of the Schrödinger Equation

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Abstract

This paper contains my version of the proof that the Schrödinger equation is covariant under the Galilean transformation. My version will follow along with the L. Ballentine and M. Weitzman versions of the proof, with minor differences.

1 Introduction

This discussion follows that presented by Leslie E. Ballentine in his book *Quantum Mechanics – A Modern Development* (1998) pp. 102–103.

The Schrödinger equation will be taken as:

$$-\frac{\hbar^2}{2M} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + W(x, t) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}, \quad (1)$$

where $W(x, t)$ is the potential function. If we take the primed system as the moving system, then the standard one-dimensional version of the Galilean transformation of coordinates are

$$x = x' + vt', \quad t = t'. \quad (2)$$

The probability of finding a particle at (x, t) in the unprimed system at physical point P at time $t = t'$ is $\Psi^*(x, t)\Psi(x, t)$, but this must be the same scalar in the primed coordinates, thus

$$\Psi^*(x, t)\Psi(x, t) = \Psi'^*(x', t')\Psi'(x', t'). \quad (3)$$

As is well known in complex number theory, we can adequately relate these two wave functions as follows:

$$\Psi(x, t) = e^{if} \Psi'(x', t'), \quad (4)$$

where f is a function of x and t , that is,

$$f = f(x, t). \quad (5)$$

For the Schrödinger equation to be covariant under change of coordinates, we need

$$-\frac{\hbar^2}{2M} \frac{\partial^2 \Psi'(x', t')}{\partial x'^2} + W'(x', t') \Psi'(x', t') = i\hbar \frac{\partial \Psi'(x', t')}{\partial t'}. \quad (6)$$

We make the assumption that $W(x, t) = W'(x', t')$.

2 The transformation of space and time derivatives

Since 1982, I have championed the subject of ‘structured differentiation’ (SD), which I invented to explain complicated differentiations to myself. Now, I will employ SD to determine how to transform $\partial/\partial x$ and $\partial/\partial t$ into primed coordinates, since we will need these derivatives to substitute into (6).

Now, let F be a scalar-valued function x, t and F' be a scalar-valued function x', t' , then

$$F'(\mathbf{x}') = F(\mathbf{x}(\mathbf{x}')), \quad (7)$$

where

$$\mathbf{x} = (x, t), \quad \mathbf{x}' = (x', t'). \quad (8)$$

Applying the chain rule for differentiation, we get

$$\frac{\partial F'}{\partial \mathbf{x}'} = \frac{\partial F}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{x}'}. \quad (9)$$

On expanding this, we get

$$\left(\frac{\partial F'}{\partial x'}, \frac{\partial F'}{\partial t'} \right) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial t} \right) \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial t'} \\ \frac{\partial t}{\partial x'} & \frac{\partial t}{\partial t'} \end{pmatrix}. \quad (10)$$

We can determine the correct values for the entries of the 2×2 matrix by differentiating the equations in (2).

$$\begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial t'} \\ \frac{\partial t}{\partial x'} & \frac{\partial t}{\partial t'} \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \quad (11)$$

where the derivatives on the LHS are all explicit, which makes them simple. Then we get, with $F' = F$:

$$\left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial t'} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \quad (12)$$

from which we have that

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \quad (13a)$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}. \quad (13b)$$

We’re going to need second derivatives by x and by x' , which, from (13a), gives us the relation

$$\frac{\partial^2}{\partial x'^2} = \frac{\partial^2}{\partial x^2}. \quad (14)$$

On making the majority of substitutions needed, we have

$$-\frac{\hbar^2}{2M} \frac{\partial^2(e^{-if}\Psi(x, t))}{\partial x^2} + W(x, t)e^{-if}\Psi(x, t) = i\hbar \frac{\partial(e^{-if}\Psi(x, t))}{\partial t'}. \quad (15)$$

3 The LHS

Let's just expand the LHS of (15) for now:

$$\begin{aligned}
& -\frac{\hbar^2}{2M} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} (e^{-if} \Psi(x, t)) + W(x, t) (e^{-if} \Psi(x, t)) \right) \\
&= -\frac{\hbar^2}{2M} \frac{\partial}{\partial x} \left(-if_x e^{-if} \Psi(x, t) + e^{-if} \frac{\partial \Psi(x, t)}{\partial x} \right) + W(x, t) (e^{-if} \Psi(x, t)) \\
&= -e^{-if} \frac{\hbar^2}{2M} \left[\left(-if_{xx} + i^2 (f_x)^2 \right) \Psi(x, t) + \left(-2if_x \frac{\partial \Psi(x, t)}{\partial x} \right) \right. \\
&\quad \left. + \frac{\partial^2 \Psi(x, t)}{\partial x^2} \right] + e^{-if} W(x, t) \Psi(x, t), \tag{16}
\end{aligned}$$

where I am using f_x and f_t as partial derivatives with respect to x and t , respectively.

4 The RHS

$$\begin{aligned}
i\hbar \frac{\partial (e^{-if} \Psi(x, t))}{\partial t} &= i\hbar \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) (e^{-if} \Psi(x, t)) \\
&= i\hbar e^{-if} \left[-if_t \Psi(x, t) + \frac{\partial \Psi(x, t)}{\partial t} + v \left\{ -if_x \Psi(x, t) + \frac{\partial \Psi(x, t)}{\partial x} \right\} \right]. \tag{17}
\end{aligned}$$

5 Combining the LHS and the RHS

On combining the LHS and the RHS and canceling out the common factor of e^{-if} , we get,

$$\begin{aligned}
& -\frac{\hbar^2}{2M} \left[\left(-if_{xx} - (f_x)^2 \right) \Psi(x, t) + \left(-2if_x(x) \frac{\partial \Psi(x, t)}{\partial x} \right) + \frac{\partial^2 \Psi(x, t)}{\partial x^2} \right] + W(x, t) \Psi(x, t) \\
&= i\hbar \left[-if_t \Psi(x, t) + \frac{\partial \Psi(x, t)}{\partial t} + v \left\{ -if_x(x) \Psi(x, t) + \frac{\partial \Psi(x, t)}{\partial x} \right\} \right]. \tag{18}
\end{aligned}$$

On rearranging and simplifying, we have that

$$\begin{aligned}
& \left\{ -\frac{\hbar^2}{2M} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + W(x, t) \Psi(x, t) - i\hbar \frac{\partial \Psi(x, t)}{\partial t} \right\} + \left[-\hbar f_t + \frac{\hbar^2}{2M} \left\{ if_{xx} + (f_x)^2 \right\} - \hbar v f_x \right] \Psi(x, t) \\
&+ \left[\frac{\hbar^2}{M} (if_x) - i\hbar v \right] \frac{\partial \Psi(x, t)}{\partial t} = 0. \tag{19}
\end{aligned}$$

On further simplification, the leftmost term on the LHS is zero because of (1). This brings us to

$$\left[-\hbar f_t + \frac{\hbar^2}{2M} \left\{ if_{xx} + (f_x)^2 \right\} - \hbar v f_x \right] \Psi(x, t) + \left[\frac{\hbar^2}{M} (if_x) - i\hbar v \right] \frac{\partial \Psi(x, t)}{\partial t} = 0. \tag{20}$$

So, we have force the coefficients of the $\Psi(x, t)$ term and the $\frac{\partial \Psi(x, t)}{\partial t}$ term to zero, but in what order? Let's start with the simpler coefficient. (One learns to do this by experience with ordinary differential equations.)

Then,

$$\frac{\hbar^2}{M} (if_x) - i\hbar v = 0. \tag{21}$$

From this we get

$$f_x = \frac{Mv}{\hbar}, \quad (22)$$

and then

$$f(x, t) = \frac{Mvx}{\hbar} + g(t), \quad (23)$$

where $g(t)$ is to be determined by our last constraint equation. Going the other way, on taking another derivative by x on (22) we get

$$f_{xx} = 0. \quad (24)$$

On setting the first coefficient to zero (and using this last equation), we get

$$-\hbar f_t + \frac{\hbar^2}{2M}(f_x)^2 - \hbar v f_x = 0. \quad (25)$$

On simplifying, we have that (with $f_t = g'(t)$)

$$-f_t + \frac{1}{2} \frac{Mv^2}{\hbar} - \frac{Mv^2}{\hbar} = 0. \quad (26)$$

This gives us

$$g(t) = -\frac{Mv^2 t}{2\hbar}. \quad (27)$$

Therefore, from (23), we have that

$$f(x, t) = \frac{Mvx}{\hbar} - \frac{Mv^2 t}{2\hbar}. \quad (28)$$