# The Gamma Matrices and Their Traces

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**Abstract**

The gamma matrices were invented by physicist Paul Dirac in his attempt to formulate a relativistic version quantum mechanics suitable for characterizing the electron. In this paper, I will focus only on the mathematical aspects of the Dirac algebra and how one uses the trace on the gamma matrices.

## 1 Introducing the gamma matrices

It's standard when introducing the gamma matrices to represent them as extension of the Pauli matrices. I won't do this here. Instead, I will introduce the four gamma matrices with lower indices, $\gamma_\mu$ where $\mu \in [\,0, 1, 2, 3\,]$. These are $4 \times 4$ matrices over the complex numbers. To these we add the unit $4 \times 4$ matrix, $I$:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1}$$

Among the four gamma matrices, one of them is not like the others. The $\gamma_0$ matrix squares to be $\gamma_0^2 = I$, where

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2}$$

Whereas, the $\gamma_i$ matrices ($i \in [1, 2, 3]$) square to be $-I$, where

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} , \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} . \tag{3}$$

The difference in these signs for the squares of these matrices represents how we've chosen our metric for the problem, though I won't go into that here.

We can already see an interesting property of the gamma matrices, which is that all their traces are zero. (See the appendices for a short tutorial on the trace function.)

The last gamma matrix which is given a standard name is $\gamma_5$, defined by

$$\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3 \,, \tag{4}$$

where $i$ is the usual imaginary unit $i = \sqrt{-1}$. Its matrix looks like:

$$\gamma_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{5}$$

Clearly, $\gamma_5$ is also traceless. Thus,

$$\mathrm{Tr}\,(\gamma_5) = 0 \,. \tag{6}$$

Now, consider the matrix formed by the product of $k$ gamma matrices. There are $\binom{4}{2} = 6$ ways to multiply two gamma matrices together (which ignores their order). A typical product of two gamma matrices, say $\gamma_1$ and $\gamma_2$, is

$$\gamma_2\gamma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} , \tag{7}$$

where $\gamma_2\gamma_1$ is traceless. Now, notice that matrix $\gamma_2\gamma_1 = -\gamma_1\gamma_2$. There are $\binom{4}{3} = 4$ ways to multiply three gamma matrices together (which also ignores their order).

It turns out that every product of the gamma matrices that does not result in a multiple of the identity matrix will have zero trace.

# 2 Further properties of the gamma matrices

The properties I want to include here are those that would have been presented already if this paper were really about the development of the Dirac equation, which it is not.

a) For $\mu, \nu = 0, 1, 2, 3$

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \quad (\mu \neq \nu) \,. \tag{8}$$

b) $\gamma_0^2 = I$, and for $i = 1, 2, 3$

$$\gamma_i^2 = -I \,. \tag{9}$$

c) $\gamma_5^2 = I$, and for $\mu = 0, 1, 2, 3$,

$$\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu \,. \tag{10}$$

d) Let's combine the information in a) and b) above to get:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = \begin{cases} 2\gamma_\mu^2 & (\mu = \nu)\,, \\ 0 & (\mu \neq \nu)\,. \end{cases} \tag{11}$$

Thus, we are in the position now to define a metric on the space of the gamma matrices:

$$g_{\mu\nu} I \equiv \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu)\,, \tag{12}$$

where the $4 \times 4$ rendering of this matrix has the form

$$g_{\mu\nu} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,. \tag{13}$$

Thus we can interpret the anticommutivity of distinct gamma matrices to imply that they are orthogonal.

**Note:** This metric we have is in flat spacetime, and so some authors use for the metric $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$.

And note that

$$\gamma_\mu \gamma^\mu = 4 \,. \tag{14}$$

We've now completed the minimum needed to define arbitrary vectors by use of the gamma matrices. Therefore, let

$$\not{a} = a^\mu \gamma_\mu \quad \text{and} \quad \not{b} = b^\nu \gamma_\nu \,, \tag{15}$$

where $a^\mu$ and $b^\nu$ are arbitrary complex numbers and we have employed the Einstein summation convention.

Show that

$$\not{a}\not{b} = 2\not{a} \cdot \not{b} I - \not{b}\not{a} \,. \tag{16}$$

First, from (12) we get that

$$\gamma_\mu \gamma_\nu = 2 g_{\mu\nu} I - \gamma_\nu \gamma_\mu \,. \tag{17}$$

Then,

$$\begin{aligned} \not{a}\not{b} &= a^\mu \gamma_\mu b^\nu \gamma_\nu \\ &= a^\mu b^\nu (2 g_{\mu\nu} I - \gamma_\nu \gamma_\mu) \\ &= 2 a^\mu b_\mu I - a^\mu b^\nu \gamma_\nu \gamma_\mu \\ &= 2\not{a} \cdot \not{b} I - \not{b}\not{a} \,. \end{aligned} \tag{18}$$

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Show that $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$.

Proof:

$$\begin{aligned}
\gamma_\mu \not{a} \gamma^\mu &= \gamma_\mu a^\alpha \gamma_\alpha \gamma^\mu \\
&= a^\alpha (2\gamma_\mu \cdot \gamma_\alpha - \gamma_\alpha \gamma_\mu) \gamma^\mu \\
&= a^\alpha (2\gamma_\mu \cdot \gamma_\alpha \gamma^\mu - \gamma_\alpha \gamma_\mu \gamma^\mu) \\
&= a^\alpha (2\gamma_\alpha - 4\gamma_\alpha) \\
&= -2\not{a} \, .
\end{aligned} \tag{19}$$

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Show that $\gamma_\mu \not{a} \not{b} \gamma^\mu = 4a \cdot b$.

Proof:

$$\begin{aligned}
\gamma_\mu \not{a} \not{b} \gamma^\mu &= (\gamma_\mu a^\alpha \gamma_\alpha) \not{b} \gamma^\mu \\
&= a^\alpha (2\gamma_\mu \cdot \gamma_\alpha - \gamma_\alpha \gamma_\mu) \not{b} \gamma^\mu \\
&= a^\alpha \not{b} (2 g_{\mu\alpha} \gamma^\mu) - a^\alpha \gamma_\alpha (\gamma_\mu \not{b} \gamma^\mu) \\
&= 2\not{b}\not{a} + 2\not{a}\not{b} \\
&= 4\not{a} \cdot \not{b} \, .
\end{aligned} \tag{20}$$

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Show that $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$.

Proof: I don't quite have the proof, but I think I have something close to it.

$$\begin{aligned}
\not{a}\not{b}\not{c} &= [\, 2\not{a} \cdot \not{b} I - \not{b}\not{a} \,] \not{c} \\
&= 2\not{a} \cdot \not{b}\not{c} - \not{b}\not{a}\not{c} \\
&= 2\not{a} \cdot \not{b}\not{c} - \not{b}[\, 2\not{a} \cdot \not{c} - \not{c}\not{a} \,] \\
&= 2\not{a} \cdot \not{b}\not{c} - 2\not{a} \cdot \not{c}\not{b} + \not{b}\not{c}\not{a} \\
&= 2\not{a} \cdot \not{b}\not{c} - 2\not{a} \cdot \not{c}\not{b} + [\, 2\not{b} \cdot \not{c} - \not{c}\not{b} ] \not{a} \\
&= 2\not{a} \cdot \not{b}\not{c} - 2\not{a} \cdot \not{c}\not{b} + 2\not{b} \cdot \not{c}\not{a} - \not{c}\not{b}\not{a} \, .
\end{aligned} \tag{21}$$

With a little manipulation, we get

$$\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a} = 2\not{a} \cdot \not{b}\not{c} - 2\not{a} \cdot \not{c}\not{b} + 2\not{b} \cdot \not{c}\not{a} \, . \tag{22}$$

Next, we apply the $\gamma_\mu \_\_ \gamma^\mu$ operator across both sides to get,

$$\gamma_\mu (\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}) \gamma^\mu = -4\not{a} \cdot \not{b}\not{c} + 4\not{a} \cdot \not{c}\not{b} - 4\not{b} \cdot \not{c}\not{a} \, . \tag{23}$$

Therefore, we can claim that

$$\gamma_\mu (\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}) \gamma^\mu = -2(\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}) \, . \tag{24}$$

It seems that the identity we want to prove follows if we can show that $\not{a}\not{b}\not{c} = \not{c}\not{b}\not{a}$, for then we would have that

$$\gamma_\mu(2\not{a}\not{b}\not{c})\gamma^\mu = -2(2\not{c}\not{b}\not{a}) \,, \tag{25}$$

or

$$\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} \,. \tag{26}$$

## 3 Trace properties of the gamma matrices

As a reminder, there are two appendices for this paper that go over the trace function/operator.

Now, in any number field, such as the real or the complex numbers, the solution to the following equation

$$x = -x \tag{27}$$

is the number zero. This 'trivial' fact comes in very handy.

Above, we showed that the trace of each the gamma matrix is zero by inspection. But, there is a more elegant way to prove this. We begin with

$$\mathrm{Tr}\,(\gamma_\mu) = \mathrm{Tr}\,(\gamma_5^2\gamma_\mu) \,, \tag{28}$$

because, as you remember, $\gamma_5^2 = I$. Thus,

$$\mathrm{Tr}\,(\gamma_\mu) = \mathrm{Tr}\,(\gamma_5\gamma_5\gamma_\mu) \,. \tag{29}$$

Now, I'm going to move the middle $\gamma_5$ to the right, using the fact that it anticommute with every gamma matrix $\gamma_\mu$. Then

$$\mathrm{Tr}\,(\gamma_\mu) = -\mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_5) \,. \tag{30}$$

But I can alternatively move the leftmost $\gamma_5$ in (29) to the right edge by using cyclic permutation, hence

$$\mathrm{Tr}\,(\gamma_\mu) = \mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_5) \,. \tag{31}$$

Therefore,

$$\mathrm{Tr}\,(\gamma_\mu) = 0 \,. \tag{32}$$

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Show that $\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = 4g_{\mu\nu}$.

First, we start with the fact that

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = \mathrm{Tr}\,(\gamma_\nu\gamma_\mu) \,. \tag{33}$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) &= \tfrac{1}{2}[\,\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) + \mathrm{Tr}\,(\gamma_\nu\gamma_\mu)\,] \\
&= \tfrac{1}{2}\mathrm{Tr}\,(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) \\
&= \mathrm{Tr}\,(\tfrac{1}{2}[\,\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\,]) \\
&= \mathrm{Tr}\,(g_{\mu\nu}I) \\
&= g_{\mu\nu}\mathrm{Tr}\,(I) \\
&= 4g_{\mu\nu}\,. \qquad (34)
\end{aligned}$$

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Show that $\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho) = 0$.

First, we start with the fact that

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho) = \mathrm{Tr}\,(\gamma_5^2\gamma_\mu\gamma_\nu\gamma_\rho) = \mathrm{Tr}\,(\gamma_5\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho)\,, \qquad (35)$$

and we proceed similarly to show we proceeded in Eq. (28). So that, when we move $\gamma_5$ to the right, transposing three times, once for each of the gamma matrices, we get

$$\mathrm{Tr}\,(\gamma_5\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho) = (-1)^3\mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_5) = -\mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_5)\,. \qquad (36)$$

On the other hand, if we start with $\mathrm{Tr}\,(\gamma_5\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho)$ and cyclicly permute the leftmost $\gamma_5$ to the rightmost spot, we get

$$\mathrm{Tr}\,(\gamma_5\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho) = \mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_5)\,. \qquad (37)$$

Hence, we conclude that

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho) = 0\,. \qquad (38)$$

And it's immediately clear by induction that this result would also hold for the trace of any odd number of gamma matrices as its argument.

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Show that

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) = 4(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\,. \qquad (39)$$

Proof:

$$\begin{aligned}
\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta &= (2g_{\mu\nu}I - \gamma_\nu\gamma_\mu)\gamma_\alpha\gamma_\beta \\
&= 2g_{\mu\nu}\gamma_\alpha\gamma_\beta - \gamma_\nu\gamma_\mu\gamma_\alpha\gamma_\beta \\
&= 2g_{\mu\nu}\gamma_\alpha\gamma_\beta - \gamma_\nu(2g_{\mu\alpha}I - \gamma_\alpha\gamma_\mu)\gamma_\beta \\
&= 2g_{\mu\nu}\gamma_\alpha\gamma_\beta - 2g_{\mu\alpha}\gamma_\nu\gamma_\beta + \gamma_\nu\gamma_\alpha\gamma_\mu\gamma_\beta \\
&= 2g_{\mu\nu}\gamma_\alpha\gamma_\beta - 2g_{\mu\alpha}\gamma_\nu\gamma_\beta + \gamma_\nu\gamma_\alpha(2g_{\mu\beta}I - \gamma_\beta\gamma_\mu) \\
&= 2g_{\mu\nu}\gamma_\alpha\gamma_\beta - 2g_{\mu\alpha}\gamma_\nu\gamma_\beta + 2g_{\mu\beta}\gamma_\nu\gamma_\alpha - \gamma_\nu\gamma_\alpha\gamma_\beta\gamma_\mu\,. \qquad (40)
\end{aligned}$$

Now we take the trace of this last equation

$$\begin{aligned}\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) &= 2g_{\mu\nu}\mathrm{Tr}\,(\gamma_\alpha\gamma_\beta) - 2g_{\mu\alpha}\mathrm{Tr}\,(\gamma_\nu\gamma_\beta) + 2g_{\mu\beta}\mathrm{Tr}\,(\gamma_\nu\gamma_\alpha) \\ &\qquad - \mathrm{Tr}\,(\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_\mu) \\ &= 8g_{\mu\nu}g_{\alpha\beta} - 8g_{\mu\alpha}g_{\nu\beta} + 8g_{\nu\alpha}g_{\mu\beta} - \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta)\,. \end{aligned} \tag{41}$$

Hence

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) = 4(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\,. \tag{42}$$

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Show that $\mathrm{Tr}\,(\gamma_5 \not{a}\not{b}) = 0$.

Proof:

First, we start with the fact that

$$\begin{aligned}\mathrm{Tr}\,(\gamma_5\not{a}\not{b}) &= \mathrm{Tr}\,[\,\gamma_5(2\not{a}\cdot\not{b}I - \not{b}\not{a})\,] \\ &= \mathrm{Tr}\,[\,\gamma_5(2\not{a}\cdot\not{b}I)\,] - \mathrm{Tr}\,(\gamma_5\not{b}\not{a}) \\ &= (2\not{a}\cdot\not{b})\mathrm{Tr}\,(\gamma_5) - \mathrm{Tr}\,(\gamma_5\not{b}\not{a}) \\ &= -\mathrm{Tr}\,(\gamma_5\not{b}\not{a})\,. \end{aligned} \tag{43}$$

Then, we can write

$$\begin{aligned}\mathrm{Tr}\,(\gamma_5\not{a}\not{b}) &= \tfrac{1}{2}[\,\mathrm{Tr}\,(\gamma_5\not{a}\not{b}) - \mathrm{Tr}\,(\gamma_5\not{b}\not{a})\,] \\ &= \tfrac{1}{2}[\,\mathrm{Tr}\,(\gamma_5(\not{a}\not{b} - \not{b}\not{a})\,]\,. \end{aligned} \tag{44}$$

Now, in plain old matrix talk, $\not{a}\not{b} - \not{b}\not{a}$ is just an antisymmetric matrix. So, let's give it the components

$$\not{a}\not{b} - \not{b}\not{a} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}\,. \tag{45}$$

On multiplying (45) on the left by $\gamma_5$, we get

$$\gamma_5(\not{a}\not{b} - \not{b}\not{a}) = \begin{pmatrix} -b & -d & 0 & f \\ -c & -e & -f & 0 \\ 0 & a & b & c \\ -a & 0 & d & e \end{pmatrix}\,, \tag{46}$$

which is clearly traceless. Hence,

$$\mathrm{Tr}\,(\gamma_5\not{a}\not{b}) = 0\,. \tag{47}$$

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Show that $\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu) = 0$.

Proof:[1] For any pair of indices $\mu, \nu$ there exists some $\alpha \neq \mu, \nu$ such that

$$\begin{aligned}\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu) &= g_{\alpha\alpha}^{-1}\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha^2)\\ &= g_{\alpha\alpha}^{-1}\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\dot{\gamma}_\alpha\gamma_\alpha)\\ &= (-1)^3 g_{\alpha\alpha}^{-1}\text{Tr}\,(\dot{\gamma}_\alpha\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha)\,,\end{aligned} \tag{48}$$

where the dotted gamma was moved to the leftmost spot by transposing it with the intervening matrices, hence, the $(-1)^3$ factor. Next,

$$\begin{aligned}\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu) &= -g_{\alpha\alpha}^{-1}\text{Tr}\,(\dot{\gamma}_\alpha\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha)\\ &= -g_{\alpha\alpha}^{-1}\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\dot{\gamma}_\alpha) \quad \text{(by cyclicly permuting)}\\ &= -g_{\alpha\alpha}^{-1}\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha^2)\\ &= -\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu)\,,\end{aligned} \tag{49}$$

Therefore, we conclude that

$$\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu) = 0\,. \tag{50}$$

Now, with this last identity, we have a more elegant way to establish (47).

$$\begin{aligned}\text{Tr}\,(\gamma_5\not{a}\not{b}) &= \text{Tr}\,(\gamma_5 a^\mu\gamma_\mu b^\nu\gamma_\nu)\\ &= a^\mu b^\nu\text{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu)\\ &= 0\,.\end{aligned} \tag{51}$$

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Show that

$$\text{Tr}\,(\not{a}\not{b}) = 4\not{a}\cdot\not{b}\,. \tag{52}$$

Proof: We begin with the fact that $\text{Tr}\,(\not{a}\not{b}) = \text{Tr}\,(\not{b}\not{a})$. Hence,

$$\begin{aligned}\text{Tr}\,(\not{a}\not{b}) &= \tfrac{1}{2}[\,\text{Tr}\,(\not{a}\not{b}) + \text{Tr}\,(\not{b}\not{a})\,]\\ &= \text{Tr}\,(\tfrac{1}{2}[\,\not{a}\not{b} + \not{b}\not{a}\,])\\ &= \text{Tr}\,(\not{a}\cdot\not{b}I)\\ &= 4\not{a}\cdot\not{b}\,.\end{aligned} \tag{53}$$

# 4 Appendix 1: The Trace Function on Matrices

This article will begin with the easy identities about the trace function and then progress to evermore difficult ones. But first, we need to define what is meant by the trace function of a square matrix.

[1]The proof I use here was inspired by a similar proof I found at https://imathworks.com/physics/physics-gamma-matrices-and-trace-operator/

**Definition:** In a square matrix $A$, the *main diagonal* starts at the upper left element $a_{11}$ and proceeds down the diagonal to the lower right $a_{nn}$. See Fig. 1.

**Definition:** The elements of $A$ not on the Main Diagonal are said to be 'off-diagonal' elements.

**Definition:** A square matrix is said to be 'diagonal' if its off-diagonal elements are all zero. Obviously, the zero matrix is trivially diagonal.

**Definition:** The symbol that will be used for the trace function in this paper is Tr (). Thus, for the $n \times n$ matrix $A$,

$$\mathrm{Tr}\,(A) \equiv a_{11} + a_{22} + \cdots + \ a_{nn}\,, \tag{54}$$

that is, the trace is the sum of the components on the main diagonal.
**Definition:** The ordered set on the elements on the Main Diagonal of matrix $A$ are presented in the convenient form $\mathrm{diag}\,(A) = (a_{11}, a_{22}, \ldots,\ a_{nn})$. (By the way, if the rows and columns start their counting at zero instead of at unity then $\mathrm{diag}\,(A) = (a_{00}, a_{11}, a_{22}, \ldots,\ a_{n-1n-1})$.) One advantage of introducing the diag() function is that it allows us to write down much more compact mathemtical expressions.

The diag() function is peculiar in that it can go the other way as well. Above, we put into its argument a matrix and received back the vector of its diagonal elements as its components. This time, we'll input a vector/array and output a diagonal matrix. Thus, for

$$v = v_1, v_2, \cdots,\ v_n\,, \tag{55}$$

then

$$\mathrm{diag}\,(v) = \begin{bmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_n \end{bmatrix}, \tag{56}$$

where, this time, the voided entries are all zeros.

If we take the composition of diag functions, $\mathrm{diag}\,(\mathrm{diag}\,(A))$, we get back a diagonal matrix $D$, having on its main diagonal the diagonal elements of $A$. Thus, for

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \tag{57}$$

then $D = \mathrm{diag}\,(\mathrm{diag}\,(A)) = \mathrm{diag}^2(A)$, and

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}. \tag{58}$$

One last comment before we begin the identities. What kind of numbers can we allow as the components of the square matrices of interest to us? Well, for many purposes, we can allow them to be just elements of a ring, meaning that they won't need to have inverses. However, for the more advances requirements, we'll see later on, the nonzero elements will need inverses. So, let's just keep things simple and assume that the components are either from the real or complex numbers.

**Definition:** An $n \times n$ matrix whose trace is zero is said to be *traceless*. Now, in a traceless matrix the components on the main diagonal need not all be zero, but if they aren't, they need to add up to zero.

**Definition:** We define the Sum() function on a vector/linear-array of numbers. Let $v$ be a vector/linear-array with $n$ components $v_1, v_2, \ldots, v_n$. Then

$$\mathrm{Sum}(v) \equiv v_1 + v_2 + \cdots + v_n \,. \tag{59}$$

The following lemma

$$\mathrm{Tr}\,(AB) \neq \mathrm{Tr}\,(A)\mathrm{Tr}\,(B)\,, \tag{60}$$

is easy to prove, by way of providing a counterexample. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{then} \quad AB = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}. \tag{61}$$

Then, $\mathrm{Tr}\,(A)\mathrm{Tr}\,(B) = 0 \cdot 2 = 0$, but $\mathrm{Tr}\,(AB) = 1$.

# 5 Appendix 2: Simple identities of the Trace function

a) Let $I_n$ be the $n \times n$ identity matrix. Then $\mathrm{Tr}\,(I_n) = n$. (Obvious.)

b) Let $A^{\mathrm{t}}$ stand for the transpose of $A$. Then $\mathrm{Tr}\,(A^{\mathrm{t}}) = \mathrm{Tr}\,(A)$. Since the transpose operation leaves the elements on the Main Diagonal fixed, this proof is obvious.

c) Let $\alpha$ be a scalar. Then $\mathrm{Tr}\,(\alpha A) = \alpha \mathrm{Tr}\,(A)$. To multiply a matrix by a scalar, one means that each element of the matrix is multiplied by the scalar. Thus,

$$\begin{aligned} \mathrm{diag}\,(\alpha A) &= (\alpha a_{11}, \alpha a_{22}, \ldots,\ \alpha a_{nn}) \\ &= \alpha(a_{11}, a_{22}, \ldots, a_{nn})\,. \end{aligned} \tag{62}$$

Now, $\mathrm{Tr}\,(\alpha A) = \mathrm{Tr}\,(\mathrm{diag}\,(\alpha A)) = \mathrm{Sum}(\mathrm{diag}\,(\alpha A)) = \alpha a_{11} + \alpha a_{22} + \cdots + \alpha a_{nn} = \alpha(a_{11} + a_{22} + \cdots + a_{nn})$. And, $\alpha \mathrm{Tr}\,(A) = \alpha \mathrm{Sum}(\mathrm{diag}\,(A)) = \alpha\,(a_{11} + a_{22} + \cdots + a_{nn})$. Hence, $\mathrm{Tr}\,(\alpha A) = \alpha \mathrm{Tr}\,(A)$.

d) Let $A, B$ be $n \times n$ matrices. Then $\mathrm{Tr}\,(A+B) = \mathrm{Tr}\,(A) + \mathrm{Tr}\,(B)$. We begin with the fact that matrices are added together component-wise, so that the $i, j$th component of $(A+B)_{ij} = A_{ij} + B_{ij}$. Therefore the $i$th component on the main diagonal of this sum is $A_{ii} + B_{ii}$. Therefore,

$$\mathrm{Tr}\,(A+B) = \sum_{i=1}^{n}(A_{ii} + B_{ii})\,. \tag{63}$$

But,

$$\mathrm{Tr}\,(A) + \mathrm{Tr}\,(B) = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = \sum_{i=1}^{n}(A_{ii} + B_{ii})\,. \tag{64}$$

Hence, $\mathrm{Tr}\,(A+B) = \mathrm{Tr}\,(A) + \mathrm{Tr}\,(B)$. Corollary: $\mathrm{Tr}\,(A-B) = \mathrm{Tr}\,(A) - \mathrm{Tr}\,(B)$.

**e)** Let $A, B$ be $n \times n$ matrices. Then $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$. The proof of this is not too difficult. The method is to look at the diagonal elements of both $AB$ and $BA$ by multiplying them together in indice form and then show that $\mathrm{diag}\,(AB) = \mathrm{diag}\,(BA)$. It's trivial from there.

**f)** Let $[\,A, B\,]$ be the commutator of $A$ and $B$, where $[\,A, B\,] \equiv AB - BA$. Show that $\mathrm{Tr}\,([\,A, B\,]) = 0$. This result follows trivially as a corollary to the last lemma.

**g)** Let $A_1, A_2, \ldots, A_k$ be $k$ $n \times n$ matrices. Then

$$\mathrm{Tr}\,(A_1 A_2 \cdots A_k) = \mathrm{Tr}\,(A_2 \cdots A_k A_1)\,. \tag{65}$$

In essence, we've cyclically permuted the left-most factor to the right side of the product. The proof of this involves induction. We need a base case to prove, which we can accept as proved by use of e), thus: $\mathrm{Tr}\,(A_1 A_2) = \mathrm{Tr}\,(A_2 A_1)$. Next, use the inductive hypothesis to assume that (65) is true for $k$ factors and then prove that the relation (65) is true for $k \to k+1$. Anticipating future needs, let's define $B = A_2 \cdots A_k A_{k+1}$, then

$$\begin{aligned}
\mathrm{Tr}\,(A_1 A_2 \cdots A_k A_{k+1}) &= \mathrm{Tr}\,(A_1 (A_2 \cdots A_k A_{k+1})) \\
&= \mathrm{Tr}\,(A_1 B) \\
&= \mathrm{Tr}\,(B A_1) \\
&= \mathrm{Tr}\,(A_2 \cdots A_k A_{k+1} A_1)\,.
\end{aligned} \tag{66}$$

Since the relation held for case $k+1$, the relation is assumed to be true for all $k \geq 2$. Now, we have shown that we can move the leftmost matrix all the way to the right, but we can also move the rightmost matrix all the way to the left by similar arguments.
**Lemma 1** (for the next theorem)

Let $A, D, P$ be $n \times n$ matrices, such that $D$ is a diagonal matrix. Suppose further that $P$ is invertible and that

$$A = P^{-1} D P\,. \tag{67}$$

Then $\mathrm{Tr}\,(A) = \mathrm{Tr}\,(D)$. We will use cyclic permutation of matrices in this proof.

Proof:

$$\mathrm{Tr}\,(A) = \mathrm{Tr}\,(P^{-1}DP) = \mathrm{Tr}\,(DPP^{-1}) = \mathrm{Tr}\,(DI) = \mathrm{Tr}\,(D)\,. \tag{68}$$

**Lemma 2** (some results without proof)

We need some result from the theory of determinants, such as, the fact that the determinant of a diagonal matrix is the product of the components on the main diagonal.

Let $A, B$ be $n \times n$ matrices, then it is known that

$$\det(AB) = \det(A)\det(B)\,. \tag{69}$$

By induction, we can show that the determinant of a product of matrices is equal to the product of the determinant of the individual matrices, or

$$\det(A_1 A_2 \cdots A_k) = \det(A_1)\det(A_2)\cdots\det(A_k)\,. \tag{70}$$

Now, if $A$ is invertible, $A^{-1}A = I$, then

$$\det(A^{-1}A) = \det(A^{-1})\det(A) = \det(I) = 1\,. \tag{71}$$