Symmetries and Wigner's Theorem

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Abstract

This paper contains my read-along notes on the lecture from Tobias Osborne from his eighteenth lecture on Advance Quantum Mechanics. [13 July 2016] The purpose of these notes is to allow the reader to follow the presentation without having to take copious notes. The fault for any inaccuracies in this paper is the author's.

1 Introduction

Why study symmetries?

With each known symmetry to a system, the fewer unknowns there are to parameterize the system. Also, known symmetries of laws help to find specific solutions to problems.

Before we get to Wigner's Theorem, we need to go over some notational matters.

Definition: A *symmetry transformation* T is a ray in Hilbert space that preserves transition probabilities. Let ϕ be a particular vector in Hilbert space $\mathscr{H}$. Let $[\,|\,\phi_1\rangle\,]$ be the set of all nonzero complex multiples of ϕ, which is a 'ray' in $\mathscr{H}$. Then we see $[\,|\,\phi_1\rangle\,]$ as an equivalence relation on Hilbert vectors such that the elements of it differ by only a complex factor — the vector $|\,\phi_1\,\rangle$ being a more or less arbitrary representative of the equivalence class.

Next, we'll put this definition into mathematical form: For all $|\,\psi_1\,\rangle \in [\,|\,\phi_1\rangle\,]$ and for all $|\,\psi_2\,\rangle \in [\,|\,\phi_2\rangle\,]$ and for all $|\,\psi_1'\,\rangle \in T([\,|\,\phi_1\rangle\,])$ and for all $|\,\psi_2'\,\rangle \in T([\,|\,\phi_2\rangle\,])$

$$\big|\,\langle\,\psi_1'\,|\,\psi_2'\,\rangle\,\big|^2 = \big|\,\langle\,\psi_1\,|\,\psi_2\,\rangle\,\big|^2\,, \tag{1}$$

where we can define a 'ray' rigorously, but in generic terms, as

$$[\,|\,\phi\rangle\,] = \{z\,|\,\phi\,\rangle \;\big|\; |\,z\,|^2 = 1,\ z \in \mathbb{C}\}\,. \tag{2}$$

For the present discussion, this definition, though not the most general, is adequate.

Note: We will make referral of (1) often throughout this paper because the transformations we are interested here are symmetry transformations and, by definition, (1) is always true in that case.

Note: It should be noted at this point that this transformation T is not proven to be linear.

2 Wigner's Theorem and its proof

Let $\mathscr{H}$ be a separable Hilbert space. Let T be any invertible symmetry transformation on $\mathscr{H}$. Then T is either unitary or anti-unitary.[1]

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[1] Time-reversal is an example of an anti-unitary transformation.

Proof:

Let $\{|\,\psi_j\,\rangle\}$ be a complete orthonormal basis for $\mathscr{H}$, where

$$|\,\psi_j\,\rangle \in [\,|\,\phi_j\rangle\,] \quad \text{with} \quad \langle\,\psi_j\,|\,\psi_k\,\rangle = \delta_{jk}\,. \tag{3}$$

Let $\big|\,\psi'_j\,\rangle \in T([\,|\,\phi_j\rangle\,])$. From (1), we get that

$$\big|\,\langle\,\psi'_j\,|\,\psi'_k\,\rangle\,\big|^2 = \big|\,\langle\,\psi_j\,|\,\psi_k\,\rangle\,\big|^2 = \delta_{jk}\,, \tag{4}$$

because

$$\langle\,\psi'_j\,|\,\psi_j\,\rangle \geq 0\,, \tag{5}$$

then

$$\langle\,\psi'_j\,|\,\psi'_k\,\rangle = \delta_{jk}\,. \tag{6}$$

From this, we know that the image vectors on the set of the original basis vectors, form an orthonormal basis for the image space, which may not be the entire Hilbert space, but our job is to show that it is. We will do this by using a proof by contradiction.

Lemma A: When a nonzero vector is expressed by expansion on a basis, it is not possible that every coefficient for each basis vectors is zero.

Lemma B: The image space of our original basis for $\mathscr{H}$ under transformation T is all also a basis for $\mathscr{H}$.

This may seem intuitively obvious, given that T is invertible, but we have to prove that the image of $\mathscr{H}$ under T is not a proper subspace of $\mathscr{H}$. We'll perform a proof by contradiction.

Lemma C: The inverse of a symmetry transformation is also a symmetry transformation.

Suppose that $\{\big|\,\psi'_j\,\rangle\}$ is not a complete basis for $\mathscr{H}$. In other words, we assume that the span of $\{\big|\,\psi'_j\,\rangle\}$ is not $\mathscr{H}$. Then there must be at least one vector, say $|\,\psi'\,\rangle$, that is orthogonal to the space of all image vectors. Then,

$$\big|\,\langle\,\psi'\,|\,\psi'_j\,\rangle\,\big|^2 = 0 \quad \forall\ j\,. \tag{7}$$

At this point, my proof will differ somewhat from Prof. Tobias's.

T operates on every vector in $\mathscr{H}$. $|\,\psi'\,\rangle$ is a vector in $\mathscr{H}$. Let $|\,\psi''\,\rangle = T^{-1}(|\,\psi'\,\rangle)$, or put another way:

$$|\,\psi'\,\rangle = T(|\,\psi''\,\rangle)\,. \tag{8}$$

Then,

$$\big|\,\langle\,\psi''\,|\,\psi_j\,\rangle\,\big|^2 = \big|\,\langle\,T^{-1}(\psi')\,|\,T^{-1}(\psi'_j)\,\rangle\,\big|^2 = \big|\,\langle\,\psi'\,|\,\psi'_j\,\rangle\,\big|^2 \equiv 0\,. \tag{9}$$

Hence,

$$\big|\,\langle\,\psi''\,|\,\psi_j\,\rangle\,\big|^2 \equiv 0\,. \tag{10}$$

Now, $|\,\psi''\,\rangle$ can always be expanded in the basis for $\mathscr{H}$, so let

$$|\,\psi''\,\rangle = \sum_k c_k\,|\,\psi_k\,\rangle\;. \tag{11}$$

Then

$$\begin{aligned} \langle\,\psi''\,|\,\psi_j\,\rangle &= \sum_k \langle\,c_k\psi_k\,|\,\psi_j\,\rangle \\ &= c_j\,\langle\,\psi_j\,|\,\psi_j\,\rangle + \sum_{k\neq j} c_k \cancelto{0}{\langle\,\psi_k\,|\,\psi_j\,\rangle} \\ &= c_j\,. \end{aligned} \tag{12}$$

Thus

$$\left|\left\langle \psi'' \middle| \psi_j \right\rangle\right|^2 = |\, c_j\,|^2 = 0 \tag{13}$$

according to (10). But this contradicts Lemma A. Hence, the assumption that the image of $\mathscr{H}$ under T does not span $\mathscr{H}$ is false.

Next, we build a phase reference because we need to know how T affects phases of basis vectors. How does T act on superpositions? WLOG, we construct the following test states:

$$|\, \xi_j \,\rangle \equiv \frac{1}{\sqrt{2}}\big(\,|\, \psi_1 \,\rangle + |\, \psi_j \,\rangle\,\big) \in [\,|\, \xi_j\rangle\,], \quad \text{for some } j \neq 1\,. \tag{14}$$

How does T act on $[\,|\, \xi_j\rangle\,]$? Let $|\, \xi_1' \,\rangle \in T([\,|\, \xi_j\rangle\,])$ and expand in terms of $\left|\, \psi_j' \,\right\rangle$. And

$$\left|\, \xi_j' \,\right\rangle = \sum_k c_{jk}\,|\, \psi_k' \,\rangle \ . \tag{15}$$

From (1) we get[2]

$$|\, c_{jj}\,| = |\, c_{j1}\,| = \frac{1}{\sqrt{2}}\,. \tag{16}$$

And for $j \neq k$, $|\, c_{jk}\,| = 0$.

For all j we can choose the phase of $|\, \xi_k' \,\rangle$ and $|\, \psi_k' \,\rangle$ so that

$$c_{jj} = c_{j1} = \frac{1}{\sqrt{2}}\,. \tag{17}$$

This phase choice is allowed since we are not also stipulating their phases after transformation.

From now on, denote for the following two vectors only (and introducing operator U)

$$\left|\, \xi_j' \,\right\rangle = U\,|\, \xi_j \,\rangle \equiv |\, U\xi_j \,\rangle \ , \tag{18a}$$

$$\left|\, \psi_j' \,\right\rangle = U\,|\, \psi_j \,\rangle \equiv |\, U\psi_j \,\rangle \ . \tag{18b}$$

Hence,

$$U\frac{1}{\sqrt{2}}(|\, \psi_j \,\rangle + |\, \psi_1 \,\rangle) = U\,|\, \xi_j \,\rangle = \frac{1}{\sqrt{2}}(U\,|\, \psi_j \,\rangle + U\,|\, \psi_1 \,\rangle)\,. \tag{19}$$

Now, let $|\, \Psi \,\rangle \in [\,|\, \Phi\rangle\,]$ be arbitrary and then expand it in the basis:

$$|\, \Psi \,\rangle = \sum_k c_j\,|\, \psi_j \,\rangle \ . \tag{20}$$

Any vector $|\, \Psi' \,\rangle$ belonging to $T([\,|\, \Phi\rangle\,])$ can be written as

$$|\, \Psi' \,\rangle = \sum_k c_j'\,\left|\, \psi_j' \,\right\rangle \ . \tag{21}$$

But we know that

$$\left|\left\langle \psi_j \middle| \Psi \right\rangle\right|^2 = \left|\left\langle U\psi_j \middle| \Psi' \right\rangle\right|^2 \ . \tag{22}$$

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[2] We cannot invoke linearity to solve for the coefficients because we have not assumed that T is linear.

Then, for all j, including $j = 1$, we have that

$$|\, c_j \,|^2 = |\, c_j' \,|^2 \,. \tag{23}$$

But we don't yet know what happened to the phases.

Additionally,

$$|\, \langle\, \xi_j |\, \Psi \,\rangle \,|^2 = |\, \langle\, U\xi_j |\, \Psi' \,\rangle \,|^2 \,, \tag{24a}$$

$$|\, c_j + c_1 \,|^2 = |\, c_j' + c_1' \,|^2 \,. \tag{24b}$$

On taking (24a)/(24b), we have that

$$(24b)/(24a) \Longrightarrow \begin{cases} \mathrm{Re}\,(c_1/c_j) = \mathrm{Re}\,(c_1'/c_j')\,, \\ \mathrm{Im}\,(c_j/c_1) = \pm \mathrm{Im}\,(c_j'/c_1')\,. \end{cases} \tag{25a}$$

Therefore

$$c_j/c_1 = c_j'/c_1' \,, \tag{26a}$$

$$c_j/c_1 = (c_j'/c_1')^* \,. \tag{26b}$$

This leaves us with four cases:

$$Uc_j = c_j' \quad \text{or} \quad c_j'^* \,, \tag{27a}$$

$$Uc_1 = c_1' \quad \text{or} \quad c_1'^* \,. \tag{27b}$$

Now, suppose that for some j,

$$c_j/c_1 = c_j'/c_1' \,, \quad \text{but for some } k \neq j \,, \tag{28a}$$

$$c_k/c_1 = c_k'^*/c_1'^* \,. \tag{28b}$$

In other words, U complex conjugates one of the coefficients but not the other, for nontrivial c_j and c_k.

Next, define

$$|\, \phi \,\rangle \equiv \frac{1}{\sqrt{3}} \big(|\, \psi_1 \,\rangle + |\, \psi_j \,\rangle + |\, \psi_k \,\rangle \big) \,. \tag{29}$$

This vector transforms to

$$|\, \phi' \,\rangle \equiv \frac{\alpha}{\sqrt{3}} \big(U \,|\, \psi_1 \,\rangle + U \,|\, \psi_j \,\rangle + U \,|\, \psi_k \,\rangle \big) \,, \tag{30}$$

where $|\, \alpha \,| = 1$.

From

$$|\, \langle\, \Phi |\, \Psi \,\rangle \,|^2 = |\, \langle\, \Phi' |\, \Psi' \,\rangle \,|^2 \,, \tag{31}$$

we need to show that

$$\mathrm{Im}\,(c_j/c_1) \cdot \mathrm{Im}\,(c_k/c_1) = 0 \,, \tag{32}$$

if they have a different behavior under the action of U Therefore, either (24a) or (24b) must hold for all j.

Contradiction! However, this special case needs to be fully generalized.

Next, define $U\,|\,\Psi\,\rangle$ to be one of the $|\,\Psi'_k\,\rangle$ belonging to transformed ray, using phase choice that

$$c_1 = c'_1 \qquad \text{or} \qquad c_1 = (c'_1)^* \,. \tag{33}$$

Then, either

$$U\big(\sum_j c_j\,|\,\psi_j\,\rangle\big) = \sum_j c_j U\,|\,\psi_j\,\rangle\;, \tag{34}$$

or

$$U\big(\sum_j c_j\,|\,\psi_j\,\rangle\big) = \sum_j c_j^* U\,|\,\psi_j\,\rangle\;. \tag{35}$$

As a possibility, the action of U could discontinuously go from (34) to (35). Suppose we find a pair $(\sum_j a_j\,|\,\psi_j\,\rangle,\ \sum_j b_j\,|\,\psi_j\,\rangle)$ in which U acts like (34) on the first, and like (35) on the second. However,

$$\Big|\sum_j b_j^* a_j\Big|^2 = \Big|\sum_j b_j a_j\Big|^2\,, \tag{36}$$

which implies that

$$\sum_{j,k} \mathrm{Im}\,(a_j^* a_k)\,\mathrm{Im}\,(b_j^* b_k) = 0\,. \tag{37}$$

Now, we can always find a third state vector $\sum_j c_j\,|\,\psi_j\,\rangle$ such that

$$\sum_{j,k} \mathrm{Im}\,(c_j^* c_k)\,\mathrm{Im}\,(a_j^* a_k) = 0\,, \tag{38}$$

and

$$\sum_{j,k} \mathrm{Im}\,(c_j^* c_k)\,\mathrm{Im}\,(b_j^* b_k) = 0\,. \tag{39}$$

Okay, so (38) implies same choice for 1st vector and third vector; and (39) implies same choice for 2nd vector and third vector.

We had to consistently use (34) and (35) across Hilbert space. Let's see what (35) means. Take two vectors

$$|\,\psi\,\rangle = \sum_j a_j\,|\,\psi_j\,\rangle \quad \& \quad |\,\phi\,\rangle = \sum_j b_j\,|\,\psi_j\,\rangle\;. \tag{40}$$

Then

$$\begin{aligned}
U(\,\alpha\,|\,\psi\,\rangle + \beta\,|\,\psi\,\rangle\,) &= U\big(\sum_j(\alpha a_j + \beta b_j)\,|\,\psi_j\,\rangle\big) \\
&= \sum_j(\alpha^* a_j^* + \beta^* b_j^*)U\,|\,\psi_j\,\rangle \\
&= \alpha^*\sum_j a_j^* U\,|\,\psi_j\,\rangle + \beta^*\sum_j b_j^* U\,|\,\psi_j\,\rangle \\
&= \alpha^* U\,|\,\psi\,\rangle + \beta^* U\,|\,\psi\,\rangle \quad (\text{use (35) again})\,.
\end{aligned} \tag{41}$$

Thus, U is either unitary or anti-unitary.

$$\begin{aligned}
\langle\,U\psi\big|\,U\phi\,\rangle &= \sum_{j,k} a_j b_k^*\,\langle\,U\psi_j\big|\,U\phi_k\,\rangle \\
&= \sum_j a_j b_j^* \\
&= \langle\,\psi_j\big|\,\phi_k\,\rangle^*\;.
\end{aligned} \tag{42}$$

And we're done with that case. Case (34) has an identical argument.