Einstein's proof of $E = mc^2$ for a photon

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Abstract

French presents his version of Einstein's version of $E = mc^2$ for a photon through a clever thought experiment.

1 Statement of the Problem

On pages 16–17 of French[1] we find a resentation of how Einstein provided a proof that photons have mass equivalence, as well as momentum and energy. We begin with the assumption that the momentum and energy of a photon are related by the equation

$$p_\gamma = \frac{E}{c}\,. \tag{1}$$

Our job here is to show that a photon has mass equivalent given by

$$m_\gamma = \frac{E}{c^2}\,. \tag{2}$$

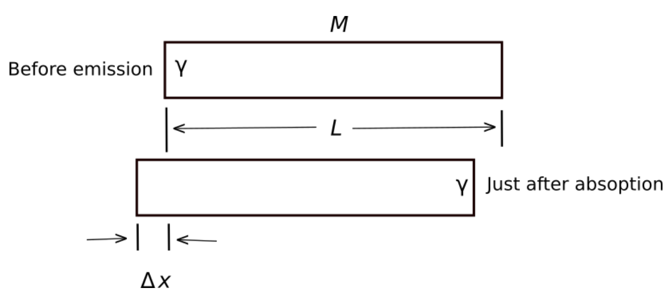

Figure 1. Photons will be emitted at the LHS and in time Δt absorbed on the RHS. When the photons are emitted, the box will recoil and move to the left at an average speed of v, going a distance Δx when it stops moving (Δx is a negative number).

[1] A.P. French, *Special Relativity*, MIT Press, Introductory Physics Series, Norton, New York, 1968.

The boxes in Fig. 1 represent the before-and-after states of a perfectly rigid box, which at some time will emit a bunch of photons at the LHS, which will then travel the length L and be absorbed at the RHS at some later time, the time interval being Δt. I am envisioning this box to rest on a perfectly frictionless surface, which means that the recoil motion to the left is stopped, not by any friction, but by the photons being absorbed at the RHS. Thus, the end-state position of the box is a little to the left of its starting point.

In a more accurate analysis of the situation, I would take into account that the photons have to travel the distance $L - \Delta x$, but since I'm assuming that Δx is much smaller than L, I'm ignoring it.

We begin our analysis with the assumption that photons have both energy E and momentum p_γ, which are related by the equation.

$$p_\gamma = E/c\,. \tag{3}$$

We also assume that the magnitude of the momentum of the photons (or a simgle photon) is equal to the magnitude of the momentum of the box in its recoil motion, or

$$p_{\text{box}} = p_\gamma\,. \tag{4}$$

2 Solution to the Problem, part 1

The time Δt, during which the box is "in motion" is the time for the photon to travel the length of the box L

$$\Delta t = \frac{L}{c}\,. \tag{5}$$

As it travels, we assume that the box moves with average speed v in the negative x direction, hence, if we assume that v is a positive number, then

$$\Delta x = -v\Delta t = -\frac{vL}{c}\,. \tag{6}$$

Now, from (4), we get

$$Mv = p_\gamma = \frac{E}{c}\,. \tag{7}$$

On removing v between these last two equations, we get

$$\Delta x = -\left(\frac{E}{Mc}\right)\frac{L}{c} = -\frac{EL}{Mc^2}\,, \tag{8}$$

which is Eq. (1-6) in the text. This equation can be rewritten as

$$M\Delta x + \left(\frac{E}{c^2}\right)L = 0\,. \tag{9}$$

Then French tells us that by considering that since the center of mass of the system should not have moved when this process is over (which is reasonable), so it follows that

$$M\Delta x + m_\gamma L = 0\,. \tag{10}$$

Then, by inspection of these last two equations, we see that

$$m_\gamma = \frac{E}{c^2}\,. \tag{11}$$

Great! So we've shown that not only do photons have energy and momentum, they also have a mass equivalence when absorbed. But this leaves us with the problem of proving (10).

3 Solution, part 2: The center of mass

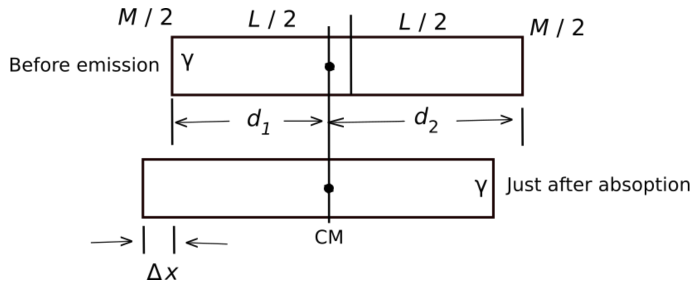

Figure 2. We suppose that the center of mass has not moved at the end of this process relative to its start. For convenience, I'm placing all the mass at the ends, equally divided between them. Consider d_1 and d_2 to be positive numbers and Δx to be a negative number, and $d_1 + d_2 = L$.

This is the hard part. First, we need one equation for the center of mass distributions for both the before-and-after states of the box. The general equation for the center of mass for a collection of two mass particles is

$$x_{\rm cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\,. \tag{12}$$

Second, we need to pay close attention to the signs of the x values.

Now, for simplicity, we will choose the "known" center of mass to be the origin of the x-axis. Thus,

$$m_1 x_1 + m_2 x_2 = 0\,. \tag{13}$$

Now, assuming that the photons have a mass equivalence, let's call it m_γ. Then, we apply this equation to both the before state and then the after state:

$$\begin{aligned} \Big(\frac{M}{2} + m_\gamma\Big)(-d_1) + \frac{M}{2} d_2 &= 0\,, \\ \frac{M}{2}\big[-(d_1 + |\,\Delta x\,|)\big] + \Big(\frac{M}{2} + m_\gamma\Big)(d_2 - |\,\Delta x\,|) &= 0\,. \end{aligned} \tag{14}$$

Next, we expand both equations. Using that fact that $m_\gamma \,|\, \Delta x \,|$ is the product of two small numbers, we'll drop this term when it arises:

$$-\frac{M}{2} d_1 - m_\gamma d_1 + \frac{M}{2} d_2 = 0\,, \tag{15a}$$

$$-\frac{M}{2} d_1 - \frac{M}{2} \,|\, \Delta x \,| + \frac{M}{2} d_2 - \frac{M}{2} \,|\, \Delta x \,| + m_\gamma d_2 = 0\,. \tag{15b}$$

Now, we subtract (15b) from (15a) to get

$$-m_\gamma d_1 + M \,|\, \Delta x \,| - m_\gamma d_2 = 0\,, \tag{16}$$

which becomes

$$m_\gamma L - M \,|\, \Delta x \,| = 0\,, \tag{17}$$

and, finally, becomes

$$m_\gamma L + M \Delta x = 0\,. \tag{18}$$

Then, on comparing this last equation with (9), we get

$$m_\gamma = \frac{E}{c^2}\,. \tag{19}$$

Well, the calculations look correct to me, and I don't know of an easier way to arrive at it. Anyway, let's not get confused here. Photons do not themselves have mass. But if they are absorbed by matter, that matter has an increase in mass according to (19).