Einstein's proof of $E = mc^2$ for a photon, redone

P. Reany

March 1, 2023

Abstract

French presents an updated version of Einstein's version of $E = mc^2$ for a photon through a clever thought experiment.

1 Statement of the Problem

On pages 27–28 of French[1] we now follow French's argument for how to put the photon mass equivalence on a better foundation. Like last time, we assumption that the momentum and energy of a photon are related by the equation

$$p_\gamma = \frac{E}{c} \,. \tag{1}$$

Our job here is to show that a photon has mass equivalent given by

$$m_\gamma = \frac{E}{c^2} \,. \tag{2}$$

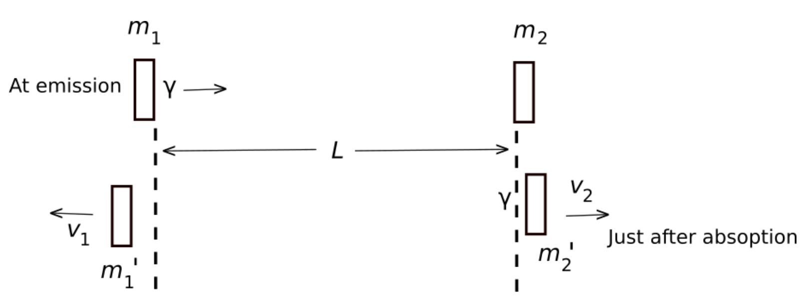

Figure 1. Photons will be emitted at the LHS at time $t = 0$ and $x = 0$. The photons emitted have total energy E. After which the box will recoil and move to the left. At time $t = L/c$, the photons are absorbed by mass m_2, which then moves to the right with mass m_2'.

[1] A.P. French, *Special Relativity*, MIT Press, Introductory Physics Series, Norton, New York, 1968.

We begin our analysis with the assumption that photons have both energy E and momentum p_γ, which are related by the equation.

$$p_\gamma = E/c\,. \tag{3}$$

Immediately after the emission on the left, the mass ($m_1' < m_1$) recoils in such a way as to preserve momentum. Thus, with new mass m_1',

$$m_1' v_1 = -p_\gamma = -\frac{E}{c}\,. \tag{4}$$

Assuming no friction, we can conclude that $v_1 = x_1/t$, so,

$$x_1 = -\frac{Et}{m_1' c}\,. \tag{5}$$

At the RHS, upon absorption of the radiant energy, we have a similar equation, expect that to measure x_2, we note that it starts at $x = L$ and at the delayed time $t - L/c$, giving us

$$x_2(t) = L + \frac{E}{m_2' c}\big(t - \frac{L}{c}\big)\,. \tag{6}$$

Once again we invoke the center-of-mass equations. Place the origin of coordinates at the right face of mass m_1 just before it emits the radiation. Also, let the total rest mass of the system be at the start

$$M = m_1 + m_2\,, \tag{7}$$

and then after absorption

$$M = m_1' + m_2'\,. \tag{8}$$

So, we have our standard equation for the center-of-mass:

$$M\overline{x} = m_1 \cdot 0 + m_2 \cdot L \tag{9}$$

at the start, and then

$$M\overline{x}' = m_1' \left(-\frac{Et}{m_1' c}\right) + m_2' \left(L + \frac{E}{m_2' c}\big(t - \frac{L}{c}\big)\right) \tag{10}$$

Now, after some simplification, we get

$$M\overline{x}' = -\frac{Et}{c} + m_2' L + \frac{E}{c}\big(t - \frac{L}{c}\big) = m_2' L - \frac{EL}{c^2}\,. \tag{11}$$

If we now insist that $M\overline{x}' = M\overline{x}$, then, using (9), we have that

$$m_2 L = m_2' L - \frac{EL}{c^2}\,. \tag{12}$$

Then, on defining $\Delta m_2 \equiv m_2' - m_2$ (the gain in mass from photon absoption), we get

$$\Delta m_2 = \frac{E}{c^2}\,. \tag{13}$$