Special Relativistic Doppler Effect 1

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Abstract

The Doppler effect of special relativity is a generalization of its predecessor in Newtonian physics. This is my first proof of the formula for relativity. It will use the least confusing proof based on SR coordinates in a Minkowski diagram.

1 Statement of the Problem

We begin with a stationary frame (S) which emits a spherical wavefront from the origin of wavelength λ. The problem is to determine the wavelength (λ') that will be seen in the moving reference frame (S'), which moves in the positive x direction at a speed v.[1]

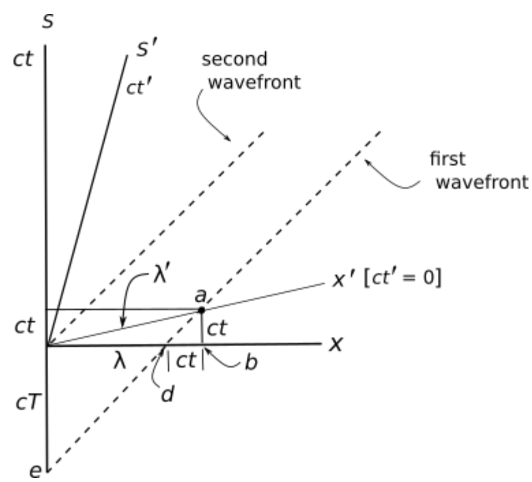

Figure 1. A wavefront is emitted at point e at time $-cT$. When that wavefront arrives at point d, the next wavefront is emitted at time 0. At time ct in frame S, the stationary frame, the first wavefront intersects the x' axis at point a. Note that on the x' axis $ct' = 0$.

[1] I will refer to the quantities ct, ct' as both a times and a distances.

Note that in the figure, the spacetime triangle abd has a lightlike side. Hence, we get that

$$c^2\tau^2 = 0 = (ct)^2 - |\,db\,|^2 \;, \tag{1}$$

from which we get that $|\,db\,| = ct$.

As usual in special relativity, we define: $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$.

The situation is depicted in Figure 1. We'll begin by stating the relevant Lorentz transformation equations. we have

$$x' = \gamma(x - vt)\,, \tag{2a}$$

$$ct' = \gamma\big(ct - \frac{vx}{c}\big)\,. \tag{2b}$$

So, we will apply these equations to the spacetime point a. In frame S,

$$a = (x, ct) = (\lambda + ct, ct)\,. \tag{3}$$

In frame S',

$$a = (x', ct') = (\lambda', 0)\,. \tag{4}$$

From this last equation and (2a), we have that

$$\lambda' = \gamma(x - vt)\,, \tag{5a}$$

and with (5b), we have that

$$0 = \gamma\big(ct - \frac{vx}{c}\big)\,. \tag{5b}$$

So, we would know λ' from (5a) if we knew both x and ct. But we can solve for both of these from the coupled equations

$$x = \lambda + ct\,, \tag{6a}$$

$$0 = ct - \frac{vx}{c}\,. \tag{6b}$$

Using (6a) in (5a), we get

$$\lambda' = \gamma[\,(\lambda + ct) - vt\,] = \gamma[\,\lambda + (c - v)t\,]\,. \tag{7}$$

On eliminating x between (6a) in (6b), we get

$$(c - v)t = \frac{v}{c}\lambda\,. \tag{8}$$

Substituting this last result into (7), we get

$$\lambda' = \gamma[\,\lambda + \frac{v}{c}\lambda\,] = \gamma[\,1 + \frac{v}{c}\,]\lambda\,. \tag{9}$$

And then finally, we have that

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1 + v/c}{1 - v/c}}\,. \tag{10}$$

The trick to this proof is to carefully set up the Minkowski diagram and then to solve the resulting algebraic jigsaw puzzle with use of the Lorentz transformation equations.