Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 10

P. Reany

May 7, 2023

Abstract

This paper contains my notes on Lecture Ten of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Energy and momentum of the EM field

My notes on this lecture begin at about time stamp 34:00 minutes.

Consider x to be multidimensional. Give $\phi(x,t)$, $\dot{\phi}$, and $\partial_x\phi$, what kind of Lagrangian can we build out of these? Beginning from classical mechanics, we have an integral over time:

$$\begin{aligned}\mathscr{A} &= \int dt L \\ &= \int dt \int d^3x \mathscr{L}\big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x^m}\big)\,. \end{aligned} \tag{1}$$

So,

$$L = \int d^3x \mathscr{L}\big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x^m}\big)\,. \tag{2}$$

Now, electromagnetism has a notion of energy associated with it. We also know that it has a momentum, which we'll refer to as Π, and we will arrive at this functional form in the standard ways of Lagrangian mechanics.

From a discrete Hamiltonian we have that

$$H = \sum_i p_i q_i - L\,. \tag{3}$$

Our starting point is the following:

$$\Pi_\phi(x) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}\,, \tag{4}$$

which allows us to define the notion of momentum that has nothing to do with the classical mv. It's referred to as *canonical momentum.*

Classically, the Hamiltonian is indexed by i as below, which is used to sum over all the degrees of freedom, but that is now replaced by the continuous variable x:

$$H = \int d^3x \big(\Pi(x)\dot{\phi}(x) - \mathscr{L}\big)\,. \tag{5}$$

Hence, we consider the quantity $\Pi_\phi(x)\dot{\phi}(x) - \mathscr{L}$ as the energy density. A possible particular Lagrangian density could be

$$\mathscr{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi)\,, \tag{6}$$

where the two terms $\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2$ is our term $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. However, we wish to keep these two terms explicit for the time being.

Using (4), we get

$$\Pi_\phi = \dot{\phi}\,. \tag{7}$$

We can now rewrite the Hamiltonian as an energy:

$$H = \int dx\big(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)\big)\,. \tag{8}$$

Let's use this 'energy' to be the time-time component T^{00} of some tensor $T^{\mu\nu}$, which we will develop as the lecture continues.

$$T^{00} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)\,. \tag{9}$$

In special relativity, the energy E is the time component of the momentum 4-vector p, and $p^0 = E$, and it is conserved locally. As before, we know that

$$j^\mu \longleftrightarrow (\rho, \mathbf{j}) \quad \text{where} \quad j^0 = \rho\,. \tag{10}$$

Then

$$\mathbf{p} = \sum p_i \delta q_i \longrightarrow \int \Pi(x)\delta\phi = \int \Pi(x)\frac{\partial\phi}{\partial x}\delta x\,. \tag{11}$$

For a simple field theory, we get a field momentum as

$$T^{0m} = \Pi\frac{\partial\phi}{\partial x^m} = \dot{\phi}\frac{\partial\phi}{\partial x^m} = \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x^m}\,. \tag{12}$$

And T^{0m} is the time component of the momentum.

Now, if we wish to enforce translational invaraince, we must demand that this expression is invariant under both particle and field translation, which will give us the conservation of momentum. T^{00} and T^{0m} are components of the energy-momentum tensor.

2 Fixing a gauge in E&M for simplicity

There are useful ways, in particular, of forming a gauge transformation

$$A_\mu \to A_\mu + \frac{\partial S}{\partial x^\mu}\,. \tag{13}$$

For example, we could choose S so that the time component of A_μ is zero. Then

$$A_0 \to A_0 + \frac{\partial S}{\partial x^0} = A_0 + \frac{\partial S}{\partial t} = 0\,, \tag{14}$$

where $\partial S/\partial t$ is at a fixed position in space.[1] Hence,

$$\frac{\partial S}{\partial t} = -A_0\,. \tag{15}$$

[1] In physics, but so usual in mathematics, the partial derivative is explicit, meaning that it does not contain contributions from changes due to motion in space (i.e., the implicit derivative part).

Thus,

$$A'_\mu = A_\mu + \frac{\partial S}{\partial x^\mu} \quad \text{with} \quad A'_0 = 0\,. \tag{16}$$

Note: We use the Maxwell convention on signs.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \cancelto{0}{\nabla A_0} = -\frac{\partial \mathbf{A}}{\partial t}\,. \tag{17}$$

And,

$$\mathbf{B} = \nabla \times \mathbf{A}\,, \tag{18}$$

is unchanged. So,

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\,. \tag{19}$$

Substituting in,

$$\mathscr{L} = \tfrac{1}{2}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 - \tfrac{1}{2}\left(\nabla \times \mathbf{A}\right)^2\,, \tag{20}$$

where $\tfrac{1}{2}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2$ acts a kinetic energy and $\tfrac{1}{2}\left(\nabla \times \mathbf{A}\right)^2$ acts as a potential energy.

What is the momentum conjugate to a particular component of the vector potential?

$$\Pi_m = \frac{\partial \mathscr{L}}{\partial(\partial_t A_m)} = \frac{\partial A_m}{\partial t} \qquad (m = 1, 2, 3)\,. \tag{21}$$

By the way,

$$\Pi_m = -E_m\,. \tag{22}$$

Now, the EM field energy is given as the Hamiltonian

$$H = \tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}\mathbf{B}^2\,. \tag{23}$$

Next, we ask is the momentum density of the EM field.

$$P = \int dx\, \Pi \frac{\partial \phi}{\partial x}\,. \tag{24}$$

Then,

$$P_n = \int dx \sum_m E_m \frac{\partial A_m}{\partial x^n}\,, \tag{25}$$

where this momentum is along the nth axis.[2]

$$\begin{aligned} \int E_m \frac{\partial A_n}{\partial x^m} d^3x &= -\int \frac{\partial E_m}{\partial x^n} d^3x + E_m A_n \Big|_{-\infty}^{+\infty} \\ &= -\int \frac{\partial E_m}{\partial x^n} d^3x \\ &= -\int \cancelto{0}{(\nabla \cdot \mathbf{E})}\, A_n\, d^3x \\ &= 0\,. \end{aligned} \tag{26}$$

[2]Thre may be a sign problem here.

Hence, (27) becomes

$$P_n = \int dx \sum_m E_m \left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m} \right) , \tag{27}$$

which takes us to the *Poynting vector*:

$$\mathbf{P} = \int \mathbf{E} \times \mathbf{B} \, d^3x \, , \tag{28}$$

which is along the direction of wave motion. (In the physics literature, the Poynting vector is most often given the symbol of a capital **S**.)

3 A bit more about the $T^{\mu\nu}$ tensor.

$T^{\mu\nu}$ is a symmetric tensor, meaning that $T^{\mu\nu} = T^{\nu\mu}$ over all indicies.

Next,

$$T^{00}, T^{01}, T^{02}, T^{03} \tag{29}$$

are densities.

$$T^{10}, T^{20}, T^{30} \tag{30}$$

represent fluxes of energy, respectively, in the x, y, z directions