Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 3

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Abstract

This paper contains my notes on Lecture Three of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Preliminaries

Binomial Theorem

$$(1+\epsilon)^p \approx 1 + p\epsilon + \frac{p(p-1)}{2!}\epsilon^2 + \cdots \,. \tag{1}$$

Thus,

$$(1-v^2)^{1/2} \approx 1 - \frac{1}{2}v^2 \,. \tag{2}$$

For classical mechanics

$$\frac{1}{(1-v^2)^{1/2}} \approx 1 + \frac{1}{2}v^2 \,. \tag{3}$$

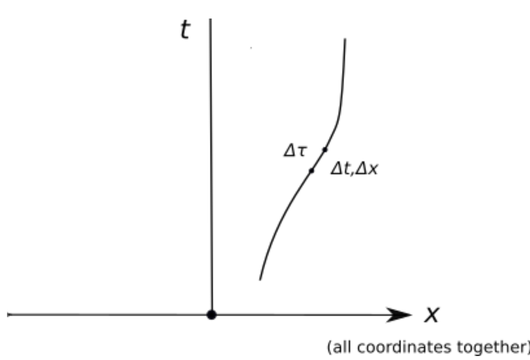

Figure 1. The trajectory of a particle is timelike.

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta \mathbf{x})^2 \,, \tag{4}$$

where

$$(\Delta\tau)^2 = \begin{cases} (\Delta t)^2 & > (\Delta \mathbf{x})^2 \quad \text{timelike}\,, \\ (\Delta t)^2 & < (\Delta \mathbf{x})^2 \quad \text{spacelike}\,, \\ (\Delta t)^2 & = (\Delta \mathbf{x})^2 \quad \text{lightlike}\,. \end{cases} \tag{5}$$

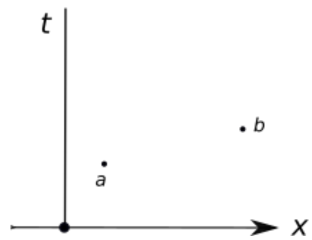

Figure 2. The interval $\overline{ab}$ is spacelike when $b_t > a_t$.

But we can always find a frame in which $b_t < a_t$, which is at the heart of the fact that distance simultaneity is frame dependent.

In the symbol Δx^μ, the greek indice will have four components, and the x^i corresponds to the latin indices i, through

$$\Delta x^\mu \sim (\Delta t, \Delta \mathbf{x})\,, \tag{6}$$

where

$$\Delta x^i \sim \Delta \mathbf{x} \qquad (i = 1, 2, 3)\,, \tag{7}$$

and where $\Delta t = \Delta x^0$.

We can arrive at $\Delta\tau$ by

$$\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta \mathbf{x})^2}\,. \tag{8}$$

One huge advantage of $\Delta\tau$ is that it is an invariant of reference frame.

Now, let's construct an invaraint notion of velocity out of the frame-dependent velocity

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \quad \text{or (in indicial notation)} \quad v^i = \frac{dx^i}{dt}\,. \tag{9}$$

So, now we construct a 4-vector:

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad (\mu = 0, 1, 2, 3)\,. \tag{10}$$

Then,

$$\begin{aligned} u^\mu = \frac{dx^\mu}{d\tau} &= (\frac{dt}{d\tau}, \frac{dx^i}{d\tau}) \\ &= (\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{dx^i}{dt}) \\ &= (\gamma, \gamma v^i) \\ &= \gamma(1, v^i)\,. \end{aligned} \tag{11a}$$

where $\gamma = 1/\sqrt{1 - v^2}$ and

$$u^0 = \gamma \equiv \frac{dt}{d\tau}, \qquad u^i = \gamma v^i\,. \tag{12}$$

One nice thing about u^μ is that it is a unit vector. Here's the proof:

$$u^\mu u_\mu = \gamma^2(1, v^i) \cdot (1, -v^i) = \gamma^2(1 - v^2) = 1\,. \tag{13}$$

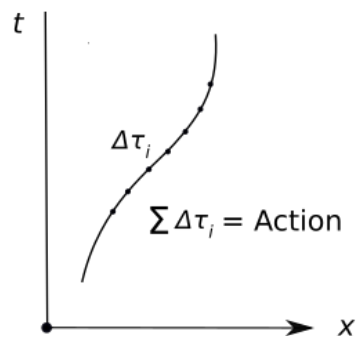

Figure 3. We define the 'action' along a timelike curve.

The following Action is defined so as to match its nonrelativistic form:

$$\text{Action} = -m \sum \Delta\tau_i\,. \tag{14}$$

Next, we go to the limit $\Delta\tau_i \to d\tau$ and then expand $d\tau$ in some particular inertial reference frame:

$$\begin{aligned}\text{Action} &= -m \int d\tau \\ &= -m \int \sqrt{dt^2 - d|\,\mathbf{x}\,|} \\ &= -m \int \sqrt{1 - v^2}\, dt \\ &= -m \int_a^b \sqrt{1 - \dot{x}^i \dot{x}^i}\, dt \quad (\text{sum on } i)\,.\end{aligned} \tag{15}$$

From this, we can determine the Lagrangian $\mathscr{L}_{fp}$ for a free particle

$$\mathscr{L}_{fp} = -m\sqrt{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} = -m\sqrt{1 - v^2}\,. \tag{16}$$

This Lagrangian is good in every reference frame. But in a reference frame in which $v << 1$, we have that

$$\mathscr{L}_{fp} = -m + \tfrac{1}{2} m v^2\,, \tag{17}$$

and, of course, we recognize $\frac{1}{2}mv^2$ as the classical kinetic energy. The m in this last equation is the so-called 'rest mass', which is the mass as measured in a frame in which the particle is at rest. But, henceforth, we shall just refer to it as the mass.

2 Momentum and Energy

Since $\mathscr{L}$ is function of velocities but not of position, then momentum is conserved.

$$\begin{aligned} p_x &= \frac{\partial \mathscr{L}}{\partial \dot{x}} \\ &= \frac{\partial}{\partial \dot{x}}\big\{ -m\sqrt{1-(\dot{x}^2+\dot{y}^2+\dot{z}^2)}\big\} \\ &= \frac{m\dot{x}}{\sqrt{1-v^2}} = \frac{mv_x}{\sqrt{1-v^2}} \\ &= \gamma m v_x \,. \end{aligned} \tag{18}$$

At this point, we have a 4-vector for momentum: $u^\mu = \gamma v^\mu$,

$$\begin{aligned} mu^x &= P_x \,, \\ mu^y &= P_y \,, \\ mu^z &= P_z \,, \\ mu^0 &= P_0 \,. \end{aligned} \tag{19}$$

Here we included the factor of c for clarity.

For us to decide what to make of p_0, we turn to the Hamiltonian.

$$H = \sum_i \dot{x}_i p^i - \mathscr{L} \,, \tag{20}$$

where

$$p^i = \frac{m\dot{x}_i}{\sqrt{1-v^2}} = \gamma m \dot{x}_i \,, \quad \text{and} \quad \mathscr{L} = -m/\gamma \,. \tag{21}$$

Then,

$$\begin{aligned} H &= \sum_i m\gamma \dot{x}_i^2 + \gamma^{-1} m \\ &= \gamma m v^2 + m\gamma^{-1} \\ &= \gamma m \,, \end{aligned} \tag{22}$$

which we identify as the energy of the particle.

But, for low velocities,

$$E \approx mc^2 + \tfrac{1}{2} m v^2 \,. \tag{23}$$

$$u^{02} - u^{x2} - u^{y2} - u^{z2} = 1 \,. \tag{24}$$

By multiplying through by m^2

$$m^2 u^{02} - m^2 u^{x2} - m^2 u^{y2} - m^2 u^{z2} = m^2 \,. \tag{25}$$

From this we get

$$E^2 - p^2 = m^2 \,. \tag{26}$$

On solving for E,

$$E = \sqrt{p^2c^2 + m^2c^4} \,. \tag{27}$$

So, what about massless particles, like the photon? In the last equation, let $m \to 0$, to get,

$$E = c|\,p\,| \,. \tag{28}$$

Let's look at positronium, the bounding together of an elelctron and a positron. Once this system collapses, the electrons annihilate each other

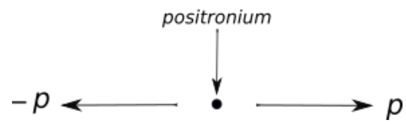

Figure 4. The unstable positronium self destructs and emits two photons in opposite directions, conserving momentum.

These two photons have the same magnitude, which we can represent as $|\,p\,|$. Mostly, the energy of the positronium is from the added masses of the two electrons. (It's actually a little less because of the 'mass defect', which is the principle that any time two particles combine into a bound state, they must sacrifice some of their combined mass.)

Anyway, the energy of one of the photons is about $m_e c^2 = h\nu$.