Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 4

P. Reany

April 12, 2023

Abstract

This paper contains my notes on Lecture Four of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Classical Field Theory

We begin with the representation of a point in spacetime:

$$x^\mu \sim (t, x^i)\,. \tag{1}$$

A field is a measureable quantity as a function of spacetime. Imagine a 1-D non-relativistic particle motion

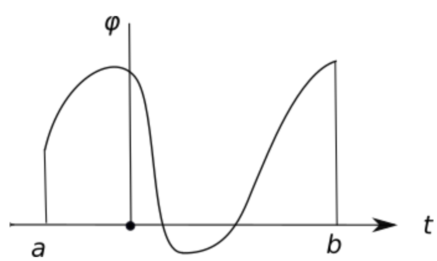

Figure 1. The representation of a field φ as a function of time.

The action is given as

$$A = \int_a^b [\,\frac{m}{2}\big(\frac{d\varphi}{dt}\big)^2 - V(\varphi)\,]dt\,, \tag{2}$$

where the integrand is our standard Lagrangian

$$\mathscr{L} = \text{KE} - \text{PE} = \frac{m}{2}\big(\frac{d\varphi}{dt}\big)^2 - V(\varphi)\,. \tag{3}$$

The equations of motion we get from this is

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\varphi}}\right) = \frac{\partial \mathscr{L}}{\partial \varphi}\,, \tag{4}$$

which reduces to

$$m\ddot{\varphi} = -\frac{\partial V}{\partial \varphi}\,. \tag{5}$$

Now, let $\varphi = \varphi(t, x^i)$. This space is referred to as (3+1) spacetime.

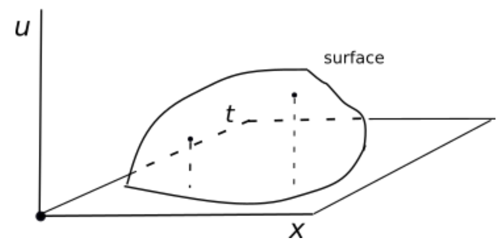

Figure 2. The two points are on the surface, which is depicted as being over the x,t-plane.

2 Lagrangian Density

Next, we generalize the Lagrangian for higher dimensions.

$$\begin{aligned} A &= \int dx\,dy\,dz\,dt\,\mathscr{L} \\ &= \int d^4x\,\mathscr{L}(\varphi, \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}) \\ &= \int d^4x\,\mathscr{L}(\varphi, \frac{\partial\varphi}{\partial x^\mu})\,. \end{aligned} \tag{6}$$

In the standard way, we arrive at the Euler-Lagrange equations:

$$A = \sum_\mu \frac{\partial}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial x^\mu}\right)} - \frac{\partial\mathscr{L}}{\partial\varphi}\,. \tag{7}$$

Now for an example:

$$\mathscr{L} = \frac{1}{2}\left[\left(\frac{\partial\varphi}{dt}\right)^2 - \left(\frac{\partial\varphi}{dx}\right)^2 - (\frac{\partial\varphi}{dy}\Big)^2 - \left(\frac{\partial\varphi}{dz}\right)^2\right] - V(\varphi)\,. \tag{8}$$

Calculating the total derivative part, we get

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t}\right) = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}\,. \tag{9}$$

So, we get

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} + \frac{\partial V}{\partial\varphi} = 0\,. \tag{10}$$

We get the wave equation in the special case that $V = 0$. In the case that $V(\varphi) = \frac{m^2}{2}\varphi^2$, we get the harmonic oscillator. Demonstrated in independent variables t, x:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2\varphi = 0\,. \tag{11}$$

This gives us wavelike oscillations, complicated by the extra term.

3 Tailoring for Special Relativity

How should we tailor the action to be consistent with special relativity? First, it's to be given as an integral over trajectories. Second, it should be constructed out of spacetime invariants, that is, to be Lorentz invariant. Third, other invariants can be formed by contracting tensors together.

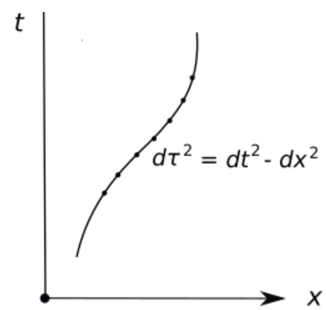

Figure 3. The main invariant alonmg a trajectory is $d\tau$.

Now, let φ be a scalar field in spacetime, as in the figure below:

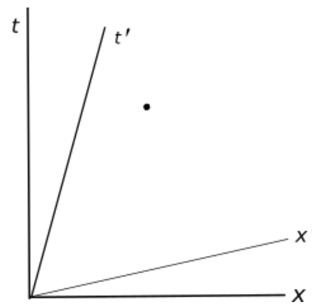

Figure 4. Both spacetime frames can indicate the point in their own coordinates. We have defined a field in the plane, called φ in the unprimed frame and called φ' in the primed frame.

So,

$$\varphi(x) = \varphi'(x') \tag{12}$$

is an example of a scalar field. Examples of vector fields are: x^μ, dx^μ, and $\frac{dx^\mu}{d\tau} = u^\mu$, where this u^μ represents an instantaneous unit vector along a spacetime trajectory.

We can write the Lorentz transformation equation in differential form as follows:

$$
\begin{aligned}
dt' &= \gamma(dt - v\,dx)\,, \\
dx' &= \gamma(dx - v\,dt)\,, \\
dy' &= dy\,, \\
dz' &= dz\,.
\end{aligned} \tag{13}
$$

You might be tempted to ask why we did not differentiate the γ. We didn't because it is only a function of the speed between the frames, and that speed is fixed for our purposes here.

We can write the transformation equations for a generic 4-vectors as follows:

$$
\begin{aligned}
(A^0)' &= \gamma(A^0 - v\,A^x)\,, \\
(A^x)' &= \gamma(A^x - v\,A^0)\,, \\
(A^y)' &= A^y\,, \\
(A^z)' &= A^z\,.
\end{aligned} \tag{14}
$$

We can change 4-vectors into scalars a few ways. For example, we can change a 4-vector into a scalar as follows:

$$
A^\mu \to (A^0)^2 - (A^x)^2 - (A^y)^2 - (A^z)^2\,. \tag{15}
$$

So, now we have some idea of sorts of things we can use to construct our Lorentz-invariant Lagrangian, which must be a scalar.

4 Constructing a Lagrangian

A general Lorentz transformation is a combination of a spacial rotation and a Lorentz boost. Let's begin with a simple action and see where it takes us.

$$
-m\int d\tau = -m\int \sqrt{dt^2 - dx^2} = -m\int dt\sqrt{1-\left(\frac{dx}{d\tau}\right)^2} = -m\int dt\sqrt{1-v^2} \tag{16}
$$

Our next step is to find a way to couple the particle to some external field φ:

One such attempt is

$$
\int (-m+\varphi)\sqrt{1-v^2}\,dt\,. \tag{17}
$$

A Higgs field can shift the mass of a particle. Anyway,

$$
\frac{\partial\mathscr{L}}{\partial\dot{x}} = [\,m-\varphi(x)\,]\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\,. \tag{18}
$$

Then,

$$
\frac{d}{dt}[\,m-\varphi(x)\,]\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} = -\frac{\partial\varphi}{\partial x}\sqrt{1-\dot{x}^2}\,. \tag{19}
$$

Which gives us

$$
-\frac{\partial\varphi}{\partial x}\frac{\dot{x}^2}{\sqrt{1-\dot{x}^2}} + (m-\varphi(x))\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} = -\frac{\partial\varphi}{\partial x}\sqrt{1-\dot{x}^2}\,. \tag{20}
$$