Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 5

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April 13, 2023

Abstract

This paper contains my notes on Lecture Five of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Relativistic Doppler Effect

The situation is depicted in Figure 1.

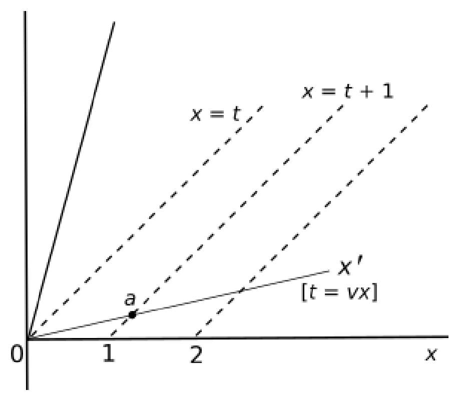

Figure 1. The spacetime diagram which will help us solve for the relativistic Doppler formula. The units along the x-axis count the number of wavelengths λ as measured in the stationary frame. Our task is reduced to finding the x' value of the point a, since $\lambda' = x'$.

We'll begin by stating the relevant Lorentz transformation equations. We have

$$x' = \gamma(x - vt)\,, \tag{1a}$$
$$t' = \gamma(t - vx)\,. \tag{1b}$$

On the x' axis, $t' = 0$, which implies from the last equation that $t = vx$, as is indicated in the figure.

So, at the intersection of the line $x = t + 1$ and $t = vx$, we get $x = vx + 1$, which can be written as

$$x = \frac{1}{1 - v}\,, \tag{2}$$

and thus

$$t = \frac{v}{1-v}\,. \tag{3}$$

So, in the stationary frame,

$$a = (x,t) = \Big(\frac{1}{1-v}, \frac{v}{1-v}\Big)\,. \tag{4}$$

In the moving frame,

$$a = (x', ct') = (\lambda', 0)\,. \tag{5}$$

From this last equation and (1a), we have that

$$\lambda' = \gamma(x - vt)\,. \tag{6a}$$

Using (2) in (3), we get

$$\lambda' = \gamma\Big[\frac{1}{1-v} - \frac{v^2}{1-v}\Big] = \gamma[1+v]\,. \tag{7}$$

And then finally, we have that

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+v}{1-v}}\,. \tag{8}$$

2 The Action Principle

We enter the subject of field theory that is concerned with the invention of an action that is Lorentz invariant and fulfils some requirement on particles and fields. The Action Principle requires that if A affects B then B affects A.

Suppose that we have a Lagrangian that is dependent on variables $x, y, \dot{x}, \dot{y}$. If we wanted a Lagrangian that decouples the x's from the y's, we could write down

$$\mathscr{L} = \mathscr{L}(x,\dot{x}) + \mathscr{L}(y,\dot{y})\,. \tag{9}$$

As an example of how to couple the x' and the y's, we could write

$$\mathscr{L} = \frac{\dot{x}}{2} + \frac{\dot{y}}{2} - V(x) - V(y) + xy\,. \tag{10}$$

Let's begin with the Lagrangian for a particle,

$$\int \mathscr{L}_{\text{part}}\,dt = \int -m\sqrt{1-\dot{x}^2}\,dt\,. \tag{11}$$

Next, we add into this the effect of some field φ:

$$\int \mathscr{L}\,dt = \int -(m + g\varphi)\sqrt{1-\dot{x}^2}\,dt\,, \tag{12}$$

where g is a parameter that acts as a coupling constant between the particle and the field. Let's put the speed of light back in so that we can find an approximation:

$$\begin{aligned}\int \mathscr{L}\,dt &= \int -(mc^2 + g\varphi)\sqrt{1-\dot{x}^2/c^2}\,dt \\ &\approx \int -(mc^2 + g\varphi)(1 - \tfrac{1}{2}\dot{x}^2)\,dt \\ &= \int (\tfrac{1}{2}m\dot{x}^2 - g\varphi)\,dt = \int (T - V)\,dt\,. \end{aligned} \tag{13}$$

Let's setup the generality at this point. We need to allow for both particle motion and field variation:

$$A_{\mathrm{F}} = \int \mathscr{L}_{\mathrm{F}}\, d^4x = \int \left[\tfrac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \tfrac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 \right] d^4x\,. \tag{14}$$

Next, we go to a special case in which we have fixed a particle at $x = 0$.

$$A_{\mathrm{part}} = \int -g\varphi(x,t)\, dt\,, \tag{15}$$

where φ reacts to the particle because it is a function of x. So, how does this change when we fix the particles at $x = 0$?

$$A_{\mathrm{part}} = \int -g\varphi(0,t)\, dt\,. \tag{16}$$

Let

$$\varphi(0,t) = \int \varphi(x,t)\delta(\mathbf{x})\, d^3x\,, \tag{17}$$

and

$$A_{\mathrm{combined}} = \int \left[\tfrac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \tfrac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 - g\,\delta^3(\mathbf{x},t) \right] d^4x\,. \tag{18}$$

From this we get the Euler-Lagrange equations:

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \frac{\partial \varphi}{\partial t}} \right) = \frac{\partial^2 \varphi}{\partial t^2}\,, \quad \text{etc.} \tag{19}$$

From this we get

$$\frac{\partial \mathscr{L}}{\partial \varphi} = \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} \tag{20}$$

or

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} = -g\,\delta^3(\mathbf{x})\,, \tag{21}$$

which is a wave equation with a point source. With the particle fixed in space, we assume that $\partial^2\varphi/\partial t^2 = 0$, hence, we have that

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = g\,\delta^3(\mathbf{x})\,, \tag{22}$$

which is Poisson's Equation. Or, expressed more tersely,

$$\nabla^2 \varphi = g\delta^3(\mathbf{x})\,. \tag{23}$$

But if the source moves, and is at $\mathbf{a}(t)$ then the equation becomes

$$-\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = g(\delta^3(\mathbf{x} - \mathbf{a}(t))\,. \tag{24}$$

3 Tensor Notation

We begin with the notation for a spacetime point

$$x^\mu = (t, x, y, z) \quad \mu = 0, 1, 2, 3\,. \tag{25}$$

A vector A^μ that transforms to the same form under a Lorentz transformation is said to be a 4-vector. And, assuming that A^μ is a 4-vector,

$$A^\mu A_\mu = {A^0}^2 - {A^1}^2 - {A^2}^2 - {A^3}^2 \tag{26}$$

is a Lorentz scalar. And we have used the Einstein Summation convention to automatically sum on any pair of indices if one is up and the other is down.

In special relativity, the metric tnesor is given in matrix form as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{27}$$

A contravariant vector A^μ is given as

$$A^\mu = \begin{pmatrix} A^t \\ A^x \\ A^y \\ A^z \end{pmatrix}\,, \tag{28}$$

We can use the metric tensor to form a covariant vector out of a contravariant vector by a process call *contraction*:

$$A_\mu = \sum \eta_{\mu\nu} A^\nu\,, \tag{29}$$

where A_μ is written as a row vector:

$$A_\mu = (-A^t, A^x, A^y, A^z)\,. \tag{30}$$

But with the Einstein summation convention, (29) becomes simply,

$$A_\mu = \eta_{\mu\nu} A^\nu\,. \tag{31}$$

Exercise: Show that $B^\mu A_\mu = A^\mu B_\mu$.

We already know that if A^μ and B^μ are arbitrary 4-vectors, that $A^\mu A_\mu$ and $B^\mu B_\mu$ are scalars. Given that, it's time to show that $A^\mu B_\mu$ is also a scalar. By similar reasoning, both $(A+B)^\mu(A+B)_\mu$ and $(A-B)^\mu(A-B)_\mu$ are scalars.

So, let's define the value X as

$$X = (A+B)^\mu(A+B)_\mu - (A-B)^\mu(A-B)_\mu\,, \tag{32}$$

which is a scalar (why?). Then,

$$\begin{aligned} X &= (A+B)^\mu(A+B)_\mu - (A-B)^\mu(A-B)_\mu \\ &= A^\mu A_\mu + A^\mu B_\mu + B^\mu A_\mu + B^\mu B_\mu - [A^\mu A_\mu - A^\mu B_\mu - B^\mu A_\mu + B^\mu B_\mu] \\ &= 2(A^\mu B_\mu + B^\mu A_\mu) \\ &= 4A^\mu B_\mu\,. \end{aligned} \tag{33}$$

Since X is a scalar, so is $A^\mu B_\mu$.

Lemma: Assuming that A_μ is a 4-vector and that $A_\mu B^\mu$ is a Lorentz scalar, show that B^μ is a 4-vector.

Proof: We have seen that for $A_\mu B^\mu$ to be a Lorentz scalar, then

$$A'_\mu B'^\mu = A^\sigma B_\sigma \,, \tag{34}$$

where A_μ has transformed as a 4-vector.

$$A'_\mu B'^\mu = L^\sigma_\mu A_\sigma B'^\mu = A_\sigma B^\sigma \,. \tag{35}$$

On rearranging, we have that

$$A_\sigma (L^\sigma_\mu B'^\mu - B^\sigma) = 0 \,. \tag{36}$$

As we're assuming that A_σ is not identically zero, by the *Quotient Rule*, we infer that

$$L^\sigma_\mu B'^\mu - B^\sigma = 0 \,, \tag{37}$$

hence that

$$B^\sigma = L^\sigma_\mu B'^\mu \,. \tag{38}$$

And this implies that B^σ has also transformed as a 4-vector.[1]

4 Differentiation

Let $\phi(x)$ be a differentiable scalar field over a region of spacetime. Further, let

$$d\phi \equiv \phi(x + dx) - \phi(x) \,. \tag{39}$$

As $d\phi$ is the difference of two scalars, we accept it as a scalar as well. Then

$$d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu \,. \tag{40}$$

Since dx^μ is a 4-vector, then $\dfrac{\partial \phi}{\partial x^\mu}$ is also a 4-vector (a covariant one).

We now introduce a common shorthand notation for differentiation:

$$\frac{\partial \phi}{\partial x^\mu} \equiv \partial_\mu \phi = \Big(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \Big) \,. \tag{41}$$

Similarly,

$$\partial^\mu \phi \equiv \frac{\partial \phi}{\partial x_\mu} = \Big(-\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \Big) \,. \tag{42}$$

For future use, we have the scalar,

$$\partial^\mu \phi \, \partial_\mu \phi = -\Big(\frac{\partial \phi}{\partial t} \Big)^2 + \Big(\frac{\partial \phi}{\partial x} \Big)^2 + \Big(\frac{\partial \phi}{\partial y} \Big)^2 + \Big(\frac{\partial \phi}{\partial z} \Big)^2 \,. \tag{43}$$

For example, consider the Lagrangian,

$$\mathscr{L} = -\tfrac{1}{2} \partial^\mu \phi \, \partial_\mu \phi \,. \tag{44}$$

[1] It doesn't matter whether we go from the primed system to the unprimed or vice versa, because the Lorentz transformation is an invertible operation.

Heuristic: Making the Lagrangian a scalar is necessary to make the integral an invariant over spacetime.

Next, we introduce the Lagrangian that's the field theoretic analog of the harmonic oscillator

$$\mathscr{L} = \tfrac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 - m^2\varphi^2\right]. \tag{45}$$

From this we get the Euler-Lagrange equations

$$\frac{d}{dx^\mu}\left(\frac{\partial\mathscr{L}}{\partial\dfrac{\partial\phi}{\partial x^\mu}}\right) - \frac{\partial\mathscr{L}}{\partial\phi} = 0\,, \tag{46}$$

giving us the Klein-Gordon equation,

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} + m^2\varphi = 0\,. \tag{47}$$

The solution of the Klein-Gordon equation is a linear combination of wave terms in the form of exponentials. So, let set

$$\phi = e^{-i\omega t}e^{(k_x x + k_y y + k_z z)} = e^{-ik_\mu x^\mu}\,, \tag{48}$$

where k_μ is a 4-vector. Then,

$$\frac{\partial^2\phi}{\partial t^2} = -m^2\phi\,, \tag{49}$$

or

$$(-\omega^2 k_x^2 + k_y^2 + k_z^2 + m^2)\phi = 0\,. \tag{50}$$

Therefore,

$$\omega = \pm\sqrt{k_x^2 + k_y^2 + k_z^2 + m^2}\,. \tag{51}$$

If needed, take the real part of ϕ. By the way, the Higgs Boson satsfies a Klein-Gordon equation.

Finally, consider the action

$$A = \int -(m+\varphi)\sqrt{1-v^2}\,. \tag{52}$$

If φ is constant, the effective mass becomes $m+\varphi$. On the other hand, if $m = 0$, φ creates 'mass' for the particle.