Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 6

P. Reany

April 17, 2023

Abstract

This paper contains my notes on Lecture Six of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Review

Consider a^μ to be a 4-vector in spacetime. Then

$$A^\mu \to A^0, A^m \quad \mu = 0,1,2,3 \quad \text{and} \quad m = 1,2,3\,. \tag{1}$$

The quantity dx^μ is a contravariant differential 4-vector. The metric of spacetime is

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2}$$

We can use this operator to create covariant 4-vector out of a contravariant 4-vector A^ν:

$$A_\mu = \eta_{\mu\nu} A^\nu \tag{3}$$

In particular:

$$A_0 = -A^0\,, \tag{4}$$

$$A_m = A^m\,. \tag{5}$$

Next, let ϕ be a scalar field. Then we can form the covariant vector

$$\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi\,. \tag{6}$$

Once again, for scalar field ϕ, we can form the differential scalar

$$d\phi = \phi(x + dx) - \phi(x) = \frac{\partial \phi}{\partial x^\mu}\, dx^\mu\,, \tag{7}$$

where $d\phi$ is given as the ocntraction of the two vectors $\dfrac{\partial \phi}{\partial x^\mu}$ and dx^μ.

If A^μ and B^μ are contravariant vectors,[1] Then $A_\mu B^\mu$ is a Lorentz scalar. Further, we can form a scalar out of B^μ by the contraction on indices: $\partial_\mu B^\mu$.

Fact: Every (proper homogeneous) Lorentz transformation can be factored as the product of a rotation and a Lorentz boost.

As a reminder $\gamma = 1/\sqrt{1-v^2} = 1/\sqrt{1-\dot{\mathbf{x}}^2}$.

Let L be a Lorentz boost at speed v along the positive x direction, then

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{8}$$

Wioth the help of this matrix, we can transform a contravariant vector A^ν in the unprimed system into the corresponding contravariant vector $A^{\mu'}$ in the primed system by

$$A^{\mu'} = L^\mu{}_\nu A^\nu . \tag{9}$$

For example, the transformation of the spacetime point x in the unprimed system is rendered in the prime system by

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} . \tag{10}$$

If the only operation we perform on coordinates is a physical rotation in the y-z plane by an angle of θ degrees, we can represent that too:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} . \tag{11}$$

$$\begin{array}{ccc} A^{\mu'} & \xrightarrow{L^\nu{}_\mu} & A^\nu \\ \eta_{\mu\rho}\big\downarrow & & \big\downarrow\eta_{\nu\tau} \\ A_\rho{}' & \xrightarrow{M^\rho{}_\tau} & A_\tau \end{array}$$

2 Tensor Notation

A scalar is a tensor of rank 0. A vector is a tensor of rank 2. Etc.

The tensor

$$T^{\mu\nu} \equiv A^\mu B^\nu \tag{12}$$

has 16 components, since both μ and ν span $0-3$ independently.

$$T^{\mu\nu'} = A^{\mu'} B^{\nu'} = L^\nu{}_\sigma L^\nu{}_\tau T^{\sigma\tau} \tag{13}$$

[1] We will always assume that vectors are 4-vectors unless explicitly stated to the contrary.

And,

$$T^{\mu\nu\lambda'} = A^{\mu'} B^{\nu'} C^{\lambda'} = L^{\nu}{}_{\sigma} L^{\nu}{}_{\tau} L^{\lambda}{}_{\rho} T^{\sigma\tau\rho} \,, \tag{14}$$

and so on.

Fact: If two tensors are equal in one frame, they must be equal in every frame. Why? Because they both transform the same way.

We have already noted that the time component of the index of a tensor is raised or lowered by the multiplication by a minus sign. For example, the first component of T^{00} is lowered by the metric by

$$T^{00} \to -T_0{}^0 \,. \tag{15}$$

And if we then lower the second component as well, we get

$$T^{00} \to T_{00} \,. \tag{16}$$

But because the space components don't change sign when raised or lowered then for and $m = 1, 2, 3$,

$$T^{0m} \to -T_{0m} \,. \tag{17}$$

It should come as no surprise then that

$$A^{\mu} B_{\nu} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \,. \tag{18}$$

Definition: If $T^{\mu\nu} = T^{\nu\mu}$, the tensor is said to be *symmetric* in its indices.

As an example,

$$T^{\mu\nu} \equiv A^{\mu} B^{\nu} + A^{\nu} B^{\mu} \tag{19}$$

is clearly symmetric.

Definition: If $T^{\mu\nu} = -T^{\nu\mu}$, the tensor is said to be *antisymmetric* in its indices.

3 Electrodynamics

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & B_2 & B_1 & 0 \end{pmatrix} . \tag{20}$$

The following is the Lorentz Force Law:

$$m\mathbf{a} = e[\,\mathbf{E} + \mathbf{v} \times \mathbf{B}\,] \,, \tag{21}$$

where e is the electric charge of the particle. Our goal is to come up with a Lagrangian that will give us a relativistic version of the Lorentz Force Law. Let's begin with a particle experiencing no external fields. Then, for starting and ending points 1 and 2, respectively, the action is givne by

$$A = \int_1^2 -m d\tau = \int_1^2 -m\sqrt{1 - \dot{\mathbf{x}}^2}\, dt \,, \tag{22}$$

where $\dot{\mathbf{x}}^2 = {\dot{x}^1}^2 + {\dot{x}^2}^2 + {\dot{x}^3}^2$.

Now, we know in advance that we can represent the EM field by a vector potential, written as $A_\mu(\mathbf{x},t)$. Let dx^μ represent a differential 4-vector along the particle trajectory. Then, $A_\mu(\mathbf{x},t)\,dx^\mu$ is a scalar, and the Action becomes

$$A_e = \int_1^2 -e\,A_\mu(\mathbf{x},t)\,dx^\mu\,, \tag{23}$$

On combining these two integrals, we get

$$\begin{aligned} A &= \int_1^2 -m\sqrt{1-\dot{\mathbf{x}}^2}\,dt - e\,A_\mu(\mathbf{x},t)\,dx^\mu \\ &= \int_1^2 [\,-m\sqrt{1-\dot{\mathbf{x}}^2} - e\,A_\mu(\mathbf{x},t)\,\dot{x}^\mu\,]\,dt\,. \end{aligned} \tag{24}$$

Now it's time to expand $A_\mu(\mathbf{x},t)\,\dot{x}^\mu$:

$$A_\mu(\mathbf{x},t)\,\dot{x}^\mu = A_0\frac{dt}{dt} + A_m(\mathbf{x},t)\,\dot{x}^m = A_0 + A_m(\mathbf{x},t)\,\dot{x}^m\,. \tag{25}$$

With this expansion, (24) becomes

$$A = \int_1^2 [\,-m\sqrt{1-\dot{\mathbf{x}}^2} - e\big(A_0(\mathbf{x},t)\,dt - eA_m(\mathbf{x},t)\,\dot{x}^m\big)\,]\,dt\,. \tag{26}$$

And that's all we need for the Action. Therefore, the integrand gives us the Lagrangian:

$$\mathscr{L} = -m\sqrt{1-\dot{\mathbf{x}}^2} - e\big(A_0(\mathbf{x},t)\,dt - eA_m(\mathbf{x},t)\,\dot{x}^m\big)\,. \tag{27}$$

So, we're now ready to construct the Euler-Lagrange equations.

$$\frac{\partial\mathscr{L}}{\partial\dot{x}^m} = m\frac{\dot{x}_m}{\sqrt{1-\dot{\mathbf{x}}^2}} - eA_m(\mathbf{x},t)\,. \tag{28}$$

We also need

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}^m}\right) = \frac{d}{dt}\Big[m\frac{m\dot{x}_m}{\sqrt{1-\dot{\mathbf{x}}^2}} - eA_m(x)\Big]\,, \tag{29}$$

and

$$\frac{\partial\mathscr{L}}{\partial x^m} = -e\frac{\partial A_0}{\partial x^m} - e\dot{x}^n\,\frac{A_n}{\partial x^m}\,. \tag{30}$$

From these we ge thte Euler-Lagrange equations:

$$m\frac{d}{dt}\frac{\dot{x}_m}{\sqrt{1-\dot{\mathbf{x}}^2}} - e\frac{\partial A_m(\mathbf{x},t)}{\partial t} - e\frac{\partial A_m(\mathbf{x},t)}{\partial x^n}\dot{x}_n = -e\frac{\partial A_0}{\partial x^m} - e\dot{x}^n\,\frac{A_n}{\partial x^m}\,. \tag{31}$$

Now for some more adjustments. We want to introduce here something like m"a":

$$m\frac{d}{dt}\frac{\dot{x}_m}{\sqrt{1-\dot{\mathbf{x}}^2}} = m\text{``}a\text{''}\,. \tag{32}$$

Then,

$$m\text{``}a\text{''} = e\left(\frac{\partial A_m(\mathbf{x},t)}{\partial x^0} - \frac{\partial A_0(\mathbf{x},t)}{\partial x^m}\right)\dot{x}_0 + e\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m}\right)\dot{x}_n\,, \tag{33}$$

where the first term on the RHS is the electric field part and the second term on the RHS is the magnetic field part.

At this point we massage the LHS.

$$m\frac{d}{dt}\frac{\dot{x}_m}{\sqrt{1-v^2}} = m\frac{\frac{dx_m}{dt}}{\sqrt{1-v^2}} = m\frac{dx_m}{d\tau}\,. \tag{34}$$

Now we let

$$u^m = \frac{dx_m}{d\tau} \quad \text{and} \quad u^0 = \frac{dx^0}{d\tau}\,. \tag{35}$$

$$m\frac{d}{d\tau}\frac{dx^m}{d\tau} = m\frac{d^2x^m}{d\tau^2} = e\left(\frac{\partial A_m(\mathbf{x},t)}{\partial x^0} - \frac{\partial A_0(\mathbf{x},t)}{\partial x^m}\right)\frac{dx^0}{d\tau} + e\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m}\right)\frac{dx^n}{d\tau}\,, \tag{36}$$

This last equation can be condensed to

$$m\frac{d}{d\tau^2}x^m = e\left(\frac{\partial A_m}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^m}\right)\frac{dx^\mu}{d\tau}\,. \tag{37}$$

Now, we already have an equation good for the space components of spacetime. We can claim that this fact, and the fact that we constructed our Action out of Lorentz-invaraint scalare, means that we can add in the equation for the time component. Therefore,

$$m\frac{d}{d\tau^2}x^\mu = e\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right)\frac{dx^\nu}{d\tau}\,. \tag{38}$$

Having thrown in the time component for free, we can write

$$\begin{cases} \dfrac{dP}{dt} = F\,, \\ dis\frac{dK}{dt} = F\cdot v\,. \end{cases} \tag{39}$$

How to impliment locality?

$$\text{Action} = \int d^4x\,\mathscr{L}(\phi,\dot{\phi})\,, \tag{40}$$

where where ϕ presents the field at the point and $\dot{\phi}$ measures the variation in the field at nearby points.

Thus, we have three principles used throughout the Standard Model and gravitational theories as guiding:

#1 Lorentz Invariance,

#2 Locality,

#3 Gauge Invariance.

This last principle will be covered in the next lecture.