Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 7

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Abstract

This paper contains my notes on Lecture Seven of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Review

Some facts about the partial derivative.

$$\partial_\mu \to (\partial_0,\, \partial_1,\, \partial_2,\, \partial_3)\,, \tag{1}$$

$$\partial^\mu \to (-\partial_0,\, \partial_1,\, \partial_2,\, \partial_3)\,. \tag{2}$$

By building the action out of Lorentz relativistic invariants, we guarantee to get the conserved quantities of physics, and that the correct equations of motion will turn up.

The points of requirement on the action A are given as:

1) Pick an action that generates the correct Euler-Lagrange equation.

2) Enforce locality condition:

$$A = \int_{\text{traj.}} dt \mathscr{L}(x, \dot{x})\,, \tag{3}$$

$$A = \int_{\text{spacetime}} dt \mathscr{L}(\varphi, \varphi_\mu)\, dx\, dy\, dz\,, \tag{4}$$

where φ is a field. An action that respects locality will not propagate superluminally across space, but tends to affects immediate neighbors in differential time.

3) Make $\mathscr{L}$ a Lorentz invariant scalar.

4) Gauge invariance.

Note: For future use: $dt/d\tau = \gamma$.

2 An action for the EM field

To write an action for the EM field, we need to introduce the vector potential A_μ:

$$\begin{aligned} \text{Action} &= \int -m d\tau - eA_\mu(x)\,dx^\mu \quad (5)\\ &= \int -m\frac{d\tau}{dt}dt - eA_\mu \frac{dx^\mu}{dt}dt \quad (6)\\ &= \int \frac{-m}{\gamma}dt - e(A_0 + A_m\dot{x}^m)\,dt\,. \quad (7)\end{aligned}$$

So, we have that

$$\mathscr{L}(t,x,\dot{x}) = \frac{-m}{\gamma} - e(A_0(\mathbf{x},t) + A_n(\mathbf{x},t)\dot{x}^n)\,, \quad (8)$$

where $\gamma = 1/\sqrt{1-v^2} = 1/\sqrt{1-\dot{\mathbf{x}}^2}$, and n runs over indices $1,2,3$.

Now we take the standard derivatives to construct the Euler-Lagrange Equations:

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\dot{x}^m} &= \frac{\partial}{\partial\dot{x}^m}\left[-m(1-\dot{x}^n\dot{x}^n)^{\frac{1}{2}} - e(A_0(\mathbf{x},t) + A_n(\mathbf{x},t)\dot{x}^n)\right]\\ &= m\gamma\dot{x}^m - eA_m(\mathbf{x},t)\\ &= m\frac{dt}{d\tau}\frac{dx^m}{dt} - eA_m(\mathbf{x},t)\\ &= m\frac{dx^m}{d\tau} - eA_m(\mathbf{x},t)\\ &= mu^m - eA_m(\mathbf{x},t)\,. \quad (9)\end{aligned}$$

On taking the total derivative by time, we have that

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}^m} &= \frac{d}{dt}\left(mu^m - eA_m\right)\\ &= m\frac{d}{dt}u^m - e\left(\frac{\partial A_m}{\partial x^0} + \frac{\partial A_m}{\partial x^n}\dot{x}^n\right) \quad (10)\end{aligned}$$

where $\partial A_m/\partial x^0$ is an explicit derivative and $(\partial A_m/\partial x^n)\dot{x}^n$ is an implicit derivative.

Continuing, we have that

$$\frac{\partial\mathscr{L}}{\partial x^m} = -e\Big(\frac{\partial A_0}{\partial x^m} + \frac{\partial A_n}{\partial x^m}\dot{x}^n\Big)\,, \quad (11)$$

and therefore

$$m\frac{d}{dt}u^m - e\left(\frac{\partial A_m}{\partial x^0} + \frac{\partial A_m}{\partial x^n}\dot{x}^n\right) = -e\Big(\frac{\partial A_0}{\partial x^m} + \frac{\partial A_n}{\partial x^m}\dot{x}^n\Big)\,. \quad (12)$$

Hence, we get a Lorentz-force type equation:

$$\begin{aligned} m\frac{d}{dt}u^m &= m\text{``}a\text{''}\\ &= e\left(\frac{\partial A_m}{\partial x^0} - \frac{\partial A_0}{\partial x^m}\right) + e\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m}\right)\dot{x}^n\\ &= e\left(E_m + (\mathbf{v}\times\mathbf{B})_m\right)\,. \quad (13)\end{aligned}$$

On multiplying through by $dt/d\tau$, we get

$$
\begin{aligned}
m\frac{du^m}{d\tau} &= e\left(\frac{\partial A_m}{\partial x^0}\frac{dx^0}{d\tau} - \frac{\partial A_0}{\partial x^m}\right)\frac{dt}{d\tau} + e\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^n}\right)\dot{x}^n\frac{dt}{d\tau} \\
&= e\left(\frac{\partial A_m}{\partial x^0} - \frac{\partial A_0}{\partial x^m}\right)\frac{dx^0}{d\tau} + e\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^n}\right)u^n \\
&= e\left(\frac{\partial A_m}{\partial x^0} - \frac{\partial A_0}{\partial x^m}\right)u^0 + e\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^n}\right)u^n\,. \qquad (14)
\end{aligned}
$$

By rewriting snd simplifying, we have that

$$
m\frac{d^2x^m}{d\tau^2} = eF^m{}_\nu u^\nu\,. \qquad (15)
$$

which has m going from 1,2,3. We can extend this equation to include the time index by the "3 implies 4" Rule to extend to include the time component, to get

$$
m\frac{d^2x^\mu}{d\tau^2} = eF^\mu{}_\nu u^\nu\,. \qquad (16)
$$

As a reminder, the EM tensor is given in matrix form as

$$
F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \qquad (17)
$$

3 Back to Lagrangians

The wave equation:

$$
\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0\,. \qquad (18)
$$

Use the Lagrangian:

$$
\mathscr{L} = -\tfrac{1}{2}\partial_\mu\partial^\mu\,. \qquad (19)
$$

What happens if $\varphi \to \varphi' = \varphi + c$, where c is an arbitrary constant? Then (18) becomes

$$
\frac{\partial^2\varphi'}{\partial t^2} - \frac{\partial^2\varphi'}{\partial x^2} = 0\,. \qquad (20)
$$

What if we had:

$$
\mathscr{L} = -\tfrac{1}{2}\partial_\mu\partial^\mu - \frac{\mu^2}{2}\varphi^2\,? \qquad (21)
$$

Then, we could not add a constant term to φ and retain the same Lagrangian. This kind of transformation of the vector portential is called a *gauge transformation*.

Fact: Adding a total derivative to a Lagrangian won't change the equations of motion.

Let's begin with the following action

$$
\mathscr{A} = -e\int A_\mu dx^\mu\,. \qquad (22)
$$

Now, let S be a scalar field. If $A_\mu \to A_\mu + \frac{\partial S}{\partial x^\mu}$ then what happens to $\mathscr{A}$?

$$\begin{aligned}
\mathscr{A} &= -e \int A_\mu dx^\mu + \frac{\partial S}{\partial x^\mu} dx^\mu \\
&= -e \int [\, A_\mu dx^\mu + S(\text{at finish point}) - S(\text{at start point})\,] \\
&= -e \int A_\mu dx^\mu + 0 \\
&= -e \int A_\mu dx^\mu \,,
\end{aligned} \tag{23}$$

since the difference in the S terms is zero during the variation.

4 Gauge Invariance

Let's show that $F_{\mu\nu}$ is gauge invariant.

$$\begin{aligned}
F'_{\mu\nu} &= \frac{\partial A'_\nu}{\partial x^\mu} - \frac{\partial A'_\mu}{\partial x^\nu} \\
&= \frac{\partial \left(A_\nu + \frac{\partial S}{\partial x^\nu} \right)}{\partial x^\mu} - \frac{\partial \left(A_\mu + \frac{\partial S}{\partial x^\mu} \right)}{\partial x^\nu} \\
&= \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial^2 S}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 S}{\partial x^\nu \partial x^\mu} \\
&= \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = F_{\mu\nu} \,,
\end{aligned} \tag{24}$$

where the double partial derivative terms cancel because we are assuming the conditions that allow for the commutation of partial derivatives.