Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 8

P. Reany

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Abstract

This paper contains my notes on Lecture Eight of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Einstein's Thought Experiment

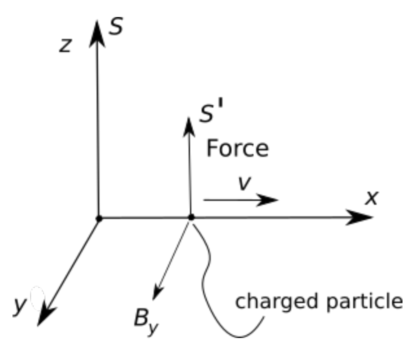

Figure 1. A charged particle at rest in S' undergoes an acceleration by an external electric field. Alternatively, if there is no electric field but there is a magnetic field as shown in S in which the particle moves at velocity v in the $+x$ direction, it will feel a force with the exact effect as the electric field alone.

Referring to Fig. 1, if we have a charged particle on the x-axis at rest in an electric field only, it will experience a force. But if we remove the electric field and replace it with a magnetic field B_y in the y-direction, and let the particle move with speed v along the x direction, then it will feel a force in the same direction as in the previous case.

2 About the $F_{\mu\nu}$ tensor

In this section we wish to analyze the $F_{\mu\nu}$ tensor to see how it relates to the electric and magnetic fields and how it relates to the Maxwell's equations, and to the vector potential A_μ. That $F_{\mu\nu}$ is antisymmetric comes from its definition:

$$F_{\mu\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\,. \tag{1}$$

So, we write down the tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{array}{c} t \\ x \\ y \\ z \end{array} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} , \tag{2}$$

where the first index is the row number and the second the column number. Let's investigate some components. For example,

$$F_{0n} = E_n = \dot{A}_n - \partial_n A_0 \,, \tag{3}$$

where n goes from $1, 2, 3$. This row represent the time-space components. This vector represents the nonzero components on the first row, and we use x, y, z and $1, 2, 3$ interchangeably.

Next, we look at the submatrix F_{mn}, which are the space-space components.

$$F_{mn} : \quad \mathbf{B} = \nabla \times \mathbf{A} \,, \tag{4}$$

or, in components:

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \,, \tag{5}$$

for an example to get B_x.

3 What about the $F^{\mu\nu}$ tensor?

We can go from the $F_{\mu\nu}$ tensor to the $F^{\mu\nu}$ tensor by separately raising each index of the former tensor. The raising operator is the metric tensor

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{6}$$

So, with the help of the Einstein summation convention and appropriate indice contraction, we get

$$\begin{aligned} F^{\alpha\beta} &= \eta^{\alpha\mu} \eta^{\beta\mu} F_{\mu\nu} \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} . \end{aligned} \tag{7}$$

4 Back to the Lorentz transformation

We use the metric tensor to transform between upper and lower indices (or vice versa) within a given reference frame. To transform a tensor between two different reference frames, we use the Lorentz transformation:[1]

$$L^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{8}$$

[1] This is not the general LT, rather, it transforms along the x, x' axes.

where v is the speed of the primed axis with respect to the unprimed axis, and $\gamma = 1/\sqrt{1-v^2}$.

So, let's perform the transformation on $F^{\mu\nu}$:

$$F^{\mu\nu\prime} = L^{\mu}{}_{\sigma} L^{\nu}{}_{\tau} F^{\sigma\tau} \,. \tag{9}$$

To illustrate, we'll focus on a particular component.

$$(E^z)' = F^{0z\prime} = L^0{}_x L^z{}_z F^{xz} = (-v\gamma)(1)B_y = -v\gamma B_y \,. \tag{10}$$

5 Identities of Maxwell's equations

Let $\mathbf{A}$ be a 3-vector. Then $\nabla \times \mathbf{A}$ is called the 'curl of $\mathbf{A}$'. Let's look at the x component of this vector.

$$(\nabla \times \mathbf{A})_x = \partial_y A_z - \partial_z A_y \,. \tag{11}$$

Reminder: For $\mathbf{B}$ a 3-vector,

$$\nabla \cdot \mathbf{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z \,. \tag{12}$$

Theorem:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \,. \tag{13}$$

Proof: Let $\mathbf{B} = \nabla \times \mathbf{A}$, then

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= \partial_x B_x + \partial_y B_y + \partial_z B_z \\
&= \partial_x (\nabla \times \mathbf{A})_x + \partial_y (\nabla \times \mathbf{A})_y + \partial_z (\nabla \times \mathbf{A})_z \\
&= \partial_x (\partial_y A_z - \partial_z A_y) + \partial_y (\partial_z A_x - \partial_x A_z) + \partial_z (\partial_x A_y - \partial_y A_x) \\
&= 0 \,,
\end{aligned} \tag{14}$$

after all the terms cancel in pairs. Hence, we have the immediate corollary that

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \,. \tag{15}$$

From (3) we have again

$$E_n = \dot{A}_n - \partial_n A_0 \,, \tag{16}$$

which becomes in 3-vector notation:

$$\mathbf{E} = \frac{\partial}{\partial t} \mathbf{A} - \nabla A_0 \,. \tag{17}$$

Now for our second identity. Let S be a differentiable function of coordinates. Then,

Theorem:

$$\nabla \times (\nabla S) = 0 \,. \tag{18}$$

Proof: We'll observe what happens to a typical component, say the x component:

$$\begin{aligned}
(\nabla \times (\nabla S))_x &= \partial_y (\nabla S)_z - \partial_z (\nabla S)_y \\
&= \partial_x \partial_z S - \partial_z \partial_y S \\
&= 0 \,,
\end{aligned} \tag{19}$$

where we have assumed that these partial derivatives commute with each other.

On taking the curl of (17), we have that

$$\begin{aligned}\nabla \times \mathbf{E} &= \nabla \times \frac{\partial}{\partial t}\mathbf{A} - \nabla \times (\nabla A_0) \\ &= \frac{\partial}{\partial t}\nabla \times \mathbf{A} - 0 \\ &= \frac{\partial}{\partial t}\mathbf{B}\,. \end{aligned} \tag{20}$$

6 Maxwell's two equations from an action principle

3) $\nabla \cdot \mathbf{E} = \rho$, where ρ is the charge density.

4) $\nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$, where $\mathbf{j}$ is the electric current density.

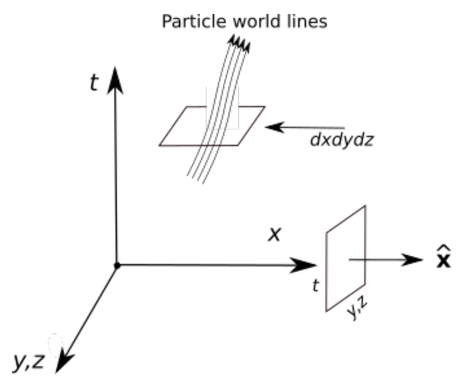

Figure 2. The amount of charge passing through a unit of physical area $\Delta A = \Delta y \Delta z$ per unit time is $j_x = \Delta Q/\Delta A \Delta t$.

With reference to the figure above, the density of charge ρ in a volume ΔV is given as

$$\rho = \frac{\Delta Q}{\Delta V}\,. \tag{21}$$

Given a small cubic volume of space, the change in charge in the cube due to the flow in the x direction in or out is the difference of the j_x values on opposite faces:

$$\delta j_x = j_x(x + \delta x, y, z) - j_x(x, y, z) = \frac{\partial j_x}{\partial x}\,\delta x\,. \tag{22}$$

Using similar results for j_y and j_z, we get the continuity equation of charge conservation:

$$\dot{\rho} = -\nabla \cdot \mathbf{j} \quad or \quad \dot{\rho} + \nabla \cdot \mathbf{j} = 0\,. \tag{23}$$

Proof: Since $\nabla \cdot \mathbf{E} = \rho$, we can write

$$\frac{\partial}{\partial t}\nabla \cdot \mathbf{E} = \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \rho}{\partial t}\,. \tag{24}$$

Now, by taking the divergence of $\nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$, we can write

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j}\,. \tag{25}$$

So,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0\,. \tag{26}$$

Advancing now to the 4-vector notation, we have the useful 4-vector related to ρ and $\mathbf{j}$ as

$$j^\mu \to (\rho, j^n) \quad \text{where} \quad n = 1,2,3 \text{ and } \mu = 0,1,2,3\,, \tag{27}$$

and with this notation (23) and (26) become

$$\frac{\partial j^\mu}{\partial x^\mu} = j^\mu{}_{,\mu} = 0\,. \tag{28}$$

7 Bianchi Identity

We have a special equation to consider now, called the *Bianchi Identity*, given by

$$\partial_\sigma F_{\nu\tau} + \partial_\nu F_{\tau\sigma} + \partial_\tau F_{\sigma\nu} = 0\,. \tag{29}$$

Let's try the combination of indice values given by, $\sigma = x\,, \nu = y\,, \tau = z$:

$$\partial_x F_{y\tau} + \partial_y F_{zx} + \partial_z F_{xy} = 0\,. \tag{30}$$

And, in terms of $\mathbf{E}$ and $\mathbf{B}$ components, we get

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0\,. \tag{31}$$

Or, more compactly,

$$\nabla \cdot \mathbf{B} = 0\,. \tag{32}$$

But what if we include a time component in the mix? $\sigma = y\,, \nu = x\,, \tau = t$:

$$\partial_y F_{xt} + \partial_x F_{ty} + \partial_t F_{yx} = 0\,. \tag{33}$$

And, in terms of $\mathbf{E}$ and $\mathbf{B}$ components, we get

$$\partial_y(-E_x) + \partial_x(E_y) + \partial_t(-B_z) = 0\,. \tag{34}$$

Or, more compactly,

$$(\nabla \times \mathbf{E})_z = (\partial_t \mathbf{B})_z\,. \tag{35}$$

A trivial proof to (29) is given by

$$\partial_\sigma \left(\frac{\partial A_\nu}{\partial x^\tau} - \frac{\partial A_\tau}{\partial x^\nu} \right) + \partial_\nu \left(\frac{\partial A_\tau}{\partial x^\sigma} - \frac{\partial A_\sigma}{\partial x^\tau} \right) + \partial_\tau \left(\frac{\partial A_\sigma}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\sigma} \right) = 0\,, \tag{36}$$

by cancellations.