Special Relativity Notes for L. Susskind's Lecture Series (2012), Lecture 9

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Abstract

This paper contains my notes on Lecture Nine of Leonard Susskind's 2012 presentation on Special Relativity for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Setting up for plane-wave solutions

First, we present three equations of the EM field:

$$\nabla \times \mathbf{A} = \mathbf{B} \tag{1a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{1b}$$
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}\,, \tag{1c}$$

Now, to investigate a plane-wave solution of these equations, we need to insist there be neither charges or currents in the vacinity. Hence

$$\nabla \cdot \mathbf{E} = 0\,, \tag{2a}$$
$$\frac{\partial \mathbf{E}}{\partial t} = -\nabla \times \mathbf{B}\,. \tag{2b}$$

Let's also remind ourselves at this point of the relationsghip between the wavelenght λ and the wavenumber k as

$$\lambda = \frac{2\pi}{k}\,. \tag{3}$$

We assume the basic ansatz solutions to be

$$E_x(z,t) = \epsilon_x(k_z - \omega t)\,, \quad B_x(z,t) = \beta_x(k_z - \omega t)\,, \tag{4a}$$
$$E_y(z,t) = \epsilon_y(k_z - \omega t)\,, \quad B_y(z,t) = \beta_y(k_z - \omega t)\,, \tag{4b}$$
$$E_z(z,t) = \epsilon_z(k_z - \omega t)\,, \quad B_z(z,t) = \beta_z(k_z - \omega t)\,, \tag{4c}$$

where the ϵ's and β's are just constants. So, we need to solve for 3 ϵ's and 3 β's. Let's solicit (2a) for some assistance.

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 0\,. \tag{5}$$

But E_x is not a function of x and E_y is not a function of y, hence we conclude that

$$\partial_z E_z = 0\,, \tag{6}$$

which forces us to conclude that $\epsilon_z = 0$. And from this we conclude that $E_z = 0$

Similarly, $\nabla \cdot \mathbf{B} = 0$, forcing the result $B_z = 0$. We can set $E_y = 0$ by a suitable rotation of axes.

From (2b), we get

$$\begin{aligned}
\frac{\partial E_x}{\partial t} &= -(\nabla \times \mathbf{B})_x \\
&= -(\partial_y B_z - \partial_z B_y) \\
&= \partial_z B_y \\
&\neq 0
\end{aligned} \tag{7}$$

since $E_x \neq 0$.

$$-\epsilon_x \omega \cos(kz - \omega t) = \beta_x k \cos(kz - \omega t)\,. \tag{8}$$

Therefore,

$$\beta_x = -\epsilon_x \frac{\omega}{k}\,. \tag{9}$$

For the y component:

$$\begin{aligned}
\frac{\partial E_y}{\partial t} &= -(\nabla \times \mathbf{B})_y \\
&= -(\partial_x B_z - \partial_z B_x)\,.
\end{aligned} \tag{10}$$

But since $E_y = 0$, then

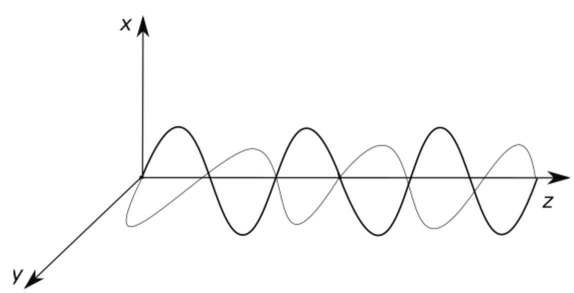
$$-(\partial_x B_z - \partial_z B_x) = 0\,. \tag{11}$$

But $B_z = 0$, so

$$\partial_z B_x = 0\,. \tag{12}$$

Thus, we are forced to set $\beta_x = 0$.

As the figure below, we choose the direction of propogation of the wave to be along the positive z axis.



The E-field is in the x, z plane, and the B-field is in the y, z plane. The E-field is perpendicular to the B-field. Together, they are transverse to their motions along the z axis.

When

$$\mathbf{E} \cdot \mathbf{B} = 0 \tag{13}$$

we say that $\mathbf{E}$ and $\mathbf{B}$ are *transverse*. Now, we have that

$$\beta = -\epsilon \frac{\omega}{k} \tag{14}$$

and also that

$$\epsilon = -\beta \frac{\omega}{k}\,. \tag{15}$$

Therefore,

$\omega/k = 1$,

$\omega = k$,

and $\epsilon = \beta$.

Hence,

$$E_x = \epsilon_x \sin k(z-t)\,, \tag{16}$$

$$B_y = \beta_y \sin k(z-t)\,. \tag{17}$$

$\nabla \cdot \mathbf{E} = \rho = j^0$, where ρ is the charge density.

$\nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$, where $\mathbf{j}$ is the electric current density.

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = j^\nu\,. \tag{18}$$

2 Back to the Lagrangians

We extremize the Action $\mathscr{A}$ of an appropriate Lagrangian L or Lagrangian density $\mathscr{L}$, when dealing with fields (possibly many of them) over a region.

$$\mathscr{A} = \int d^4x \mathscr{L}(\phi, \frac{\partial \phi}{\partial x^\mu})\,. \tag{19}$$

Making the Lagrangian density a function of $\partial\phi/\partial x^\mu$ enforces *locality* on the system. We can, also adopt a convenient notation.

$$\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi = \phi_{,\mu}\,. \tag{20}$$

Next, we need ensure that the Lagrangian is a Lorentz-invariant scalar, like

$$\mathscr{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\,. \tag{21}$$

The equations of motion are derived from the Euler-Lagrange equations:

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathscr{L}}{\partial \phi_{,\mu}} = \frac{\partial \mathscr{L}}{\partial \phi}\,, \tag{22}$$

yielding

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = -\frac{\partial V}{\partial \phi}\,. \tag{23}$$

We also need gauge invariance. (We will set $j^\mu = 0$: no charges.) So, we choose fields such $F_{\mu\nu}$ and whatever scalar you can construct with it. For gauge invariance, we need scalar forms that are invariant under the following transformation

$$A_\mu \to A'_\mu = A_\mu + \frac{\partial S}{\partial x^\mu}\,, \tag{24}$$

where S is a scalar field. Under such a transformation, the scalar $A_\mu A^\mu$ would not transform to $A'_\mu A^{\mu\prime}$.

We need to construct our scalars out of $F_{\mu\nu}$ somehow. What if we took the trace[1] of $F_\mu^\nu = F_\mu^\mu$? If we did this, the result is zero because every entry on the diagonal is zero.

However, $F_{\mu\nu}F^{\mu\nu}$ is an example of a useable scalar:

$$F_{\mu\nu}F^{\mu\nu} = -2\mathbf{E}^2 + 2\mathbf{B}^2\,. \tag{25}$$

However, by convention, we choose the form instead

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\,. \tag{26}$$

Let's adopt another convenient notation.

$$A_{\mu,\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu}\,. \tag{27}$$

Then,[2]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}\,. \tag{28}$$

So, a possible Lagrangian is

$$\frac{\partial \mathscr{L}}{\partial A_{x,y}} = -\tfrac{1}{4}(A_{\nu,\mu} - A_{\nu,\mu})(A^{\nu,\mu} - A^{\nu,\mu})\,. \tag{29}$$

We're now at the point of having to calculate a term 0f $\partial\mathscr{L}/\partial A_{\mu\nu}$. Let's take $\mu = x$ and $\nu = y$:

$$\begin{aligned} \mathscr{L}\Big|_{\substack{\mu=x\\ \nu=y}} &= -\tfrac{1}{2}\Big(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\Big)^2 \\ &= -\tfrac{1}{2}\Big(A_{x,y}^2 - 2A_{x,y}A_{y,x} + A_{y,x}^2\Big)\,. \end{aligned} \tag{30}$$

But for our purposes, $A_{x,y} \neq A_{y,x}$ in general. Therefore,

$$\frac{\partial \mathscr{L}}{\partial A_{x,y}} = -A_{x,y} + A_{y,x} = -F_{xy} = -F^{xy}\,. \tag{31}$$

Generalizing,

$$\frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}} = -F^{\mu\nu}\,. \tag{32}$$

On differentiating by $\partial/\partial x^\nu$, we get

$$\frac{\partial}{\partial x^\nu}\frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}} = -\frac{\partial F^{\mu\nu}}{\partial x^\nu}\,. \tag{33}$$

Now,

$$\frac{\partial}{\partial x^\nu}\frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}} = \frac{\partial \mathscr{L}}{\partial A_\mu}\,. \tag{34}$$

But $\mathscr{L}$ is not a function of A_μ, therefore,

$$\frac{\partial \mathscr{L}}{\partial A_\mu} = 0\,, \tag{35}$$

[1] The *trace* of a square matrix is the sum of all its diagonal elements.

[2] I'm using the definition of $F_{\mu\nu}$ from Wikipedia. Anyway, since we will in essense be taking a 'square' in the F tensor to form the Lagrangian, it won't matter.

and therefore,

$$\frac{\partial}{\partial x^\nu}\frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}} = 0\,. \tag{36}$$

And, finally, we get the important result that

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0\,. \tag{37}$$

3 What if we have charge density and charge current?

We begin with the action

$$\mathscr{A} = -\int d^4x j^\mu(x)\,A_\mu(x)\,. \tag{38}$$

After doing a gauge transformation, with the assumption that $j^\mu \to 0$ at infinity, we get

$$\mathscr{A}' = -\int d^4x j^\mu(x)\,A_\mu - \int d^4x j^\mu(x)\,\frac{\partial S}{\partial x^\mu}\,. \tag{39}$$

If we can sahow that the second term in the last equation is zero, then we will have shown that the action $\mathscr{A}$ is invariant under a gauge transformation. To that end we let

$$\mathscr{A}_g \equiv -\int d^4x j^\mu(x)\,\frac{\partial S}{\partial x^\mu}\,. \tag{40}$$

and seek to show that $\mathscr{A}_g = 0$. Let's look closer at the integrand, namely

$$j^0\,\frac{\partial S}{\partial x^0} + j^1\,\frac{\partial S}{\partial x^1} + j^2\,\frac{\partial S}{\partial x^2} + j^3\,\frac{\partial S}{\partial x^3}\,. \tag{41}$$

Furthermore, let's examine one of these terms to see how it integrates. So, let I_x be defined as

$$I_x = \int j^x\,\frac{\partial S}{\partial x}\,dx\,dy\,dz\,dt\,. \tag{42}$$

Now,

$$\partial_x(j^x S) = (\partial_x j^x)S + j^x\,\frac{\partial S}{\partial x} = (\partial_x j^x)S + j^x\,\partial_x S\,. \tag{43}$$

Similarly,

$$\partial_y(j^y S) = (\partial_y j^y)S + j^y\,\frac{\partial S}{\partial y} = (\partial_y j^y)S + j^y\,\partial_y S\,, \tag{44}$$

and

$$\partial_z(j^z S) = (\partial_z j^z)S + j^z\,\frac{\partial S}{\partial z} = (\partial_z j^z)S + j^z\,\partial_z S\,. \tag{45}$$

Adding these up gives

$$\partial_x(j^x S) + \partial_y(j^y S) + \partial_z(j^z S) = (\nabla\cdot\mathbf{j})S + \mathbf{j}\cdot\nabla S\,. \tag{46}$$

Similarly,

$$\partial_t(j^t S) = (\partial_t j^t)S + j^t\,\frac{\partial S}{\partial t} = (\partial_t j^t)S + j^t\,\partial_t S\,. \tag{47}$$

On subtracting (47) from (46), we have that

$$\begin{aligned} -\partial_t(j^t S) + \partial_x(j^x S) + \partial_y(j^y S) + \partial_z(j^z S) &= \cancelto{0}{(\partial_\mu j^\mu)}S + j^\mu\partial_\mu S \\ \text{thus,}\quad \partial_\mu(j^\mu S) &= j^\mu\partial_\mu S\,. \end{aligned} \tag{48}$$

Now, substituting this last result into (40), we get

$$\mathscr{A}_g = -\int d^4x\,\partial_\mu(j^\mu S) = -j^\mu S\big|_{\substack{\text{spacetime}\\ \text{at infinity}}} = 0\,, \tag{49}$$

because we have assumed that $j^\mu(x)$ goes to zero on the infinite boundaries. Thus we have shown that the term $j^\mu A_\mu$ in the Lagrangian is gauge invariant.

So, we set

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \quad \text{with} \quad \partial_\mu j^\mu = 0\,, \tag{50}$$

and

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu\,. \tag{51}$$

And, finally, a consistency check.

$$\frac{\partial}{\partial x^\mu}\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu}F^{\mu\nu}\,. \tag{52}$$

However,

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu}\,. \tag{53}$$

is symmetric in μ and ν, whereas, $F^{\mu\nu}$ is antisymmetric in them; hence, the summation on them results in zero. From this, we get back $\partial_\mu j^\mu = 0$.