String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 2

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Abstract

This paper contains my notes on Lecture Two of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belong solely to me.

1 A Foundation in mathematics

Basic stuff in calculus.

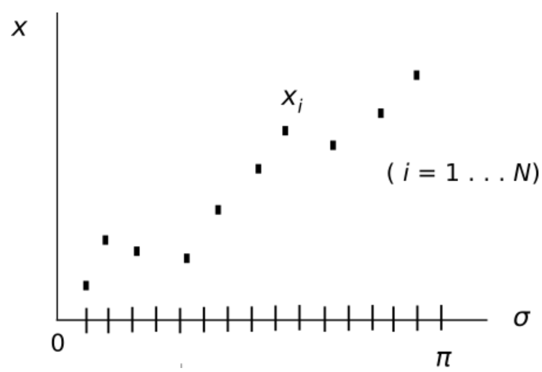

Figure 1. Going from the discrete to the continuous. Variables x and σ are the standards.

$$\Delta x \equiv x_i - x_{i-1} \,. \tag{1}$$

We are regarding x as a function, and we assume that it is differentiable.

$$\Delta x \approx \frac{\partial x}{\partial \sigma} \Delta \sigma = \frac{\partial x}{\partial \sigma} \frac{\pi}{N} \,. \tag{2}$$

$$\pi = (N) \frac{\pi}{N} = N \Delta \sigma \,. \tag{3}$$

$$\Delta \sigma \sum x_i \to \int_0^\pi x(\sigma) \, d\sigma \,. \tag{4}$$

Note: Any continuous function on 0 to π can be expanded in sines and cosines. This brings us to the important issue of boundary conditions (BC). We have two of specific interest:

Dirichlet B.C. or Neumann B.C. For Dirichlet, we have $x(0) = 0$ and $x(\pi) = 0$. For Neumann BC, we have $\dfrac{\partial x(0)}{\partial \sigma} = 0$ and $\dfrac{\partial x(\pi)}{\partial \sigma} = 0$.

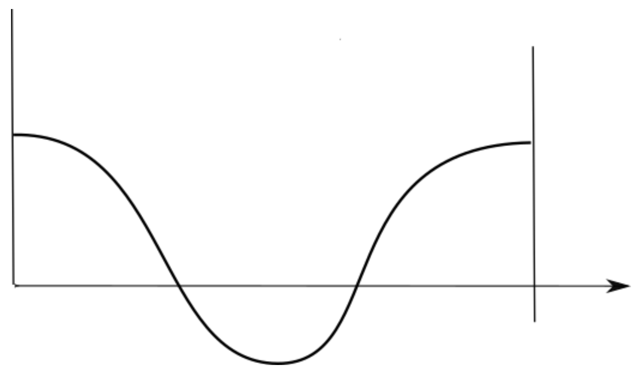

Figure 2. Going from the discrete to the continuous. This is a Neumann version.

The Fourier expansion of the Dirichlet BC:

$$x(\sigma) = \sum_{n=1}^{\infty} x_n \sin n\sigma \,. \tag{5}$$

The sine is zero at $\sigma = 0$, $\sigma = \pi$

Neumann B.C.

$$x(\sigma) = \sum_{n=0}^{\infty} x_n \cos n\sigma \,, \tag{6}$$

$$\dot{x}(\sigma) = 0 \text{ at } \sigma = 0\,,\ \ \sigma = \pi \,. \tag{7}$$

For $n, m \geq 0$

$$I = \int_0^{\pi} \cos n\sigma \cos m\sigma \, d\sigma \,. \tag{8}$$

$$\cos n\sigma \cos m\sigma \ = \tfrac{1}{2}(\cos \sigma(n-m) + \cos \sigma(n+m) \,. \tag{9}$$

$$I = \tfrac{1}{2} \int_0^{\pi} \big[\cos \sigma(n-m) + \cos \sigma(n+m) \big] \, d\sigma \,. \tag{10}$$

$$I = \tfrac{1}{2} \int_0^{\pi} \cos^2 \sigma n = \frac{\pi}{2} \,. \tag{11}$$

$$\int_0^{\pi} \cos n\sigma \cos m\sigma \, d\sigma = \frac{\pi}{2} \delta_{nm} \,, \quad \text{if } n \neq 0, m \neq 0 \,, \tag{12}$$

with special case

$$I = \int_0^\pi \cos n\sigma \cos m\sigma \, d\sigma = \pi \,, \quad \text{if } n = m = 0 \,. \tag{13}$$

For $n = 0$,

$$\int_0^\pi \cos m\sigma \, d\sigma = \frac{-1}{m} \sin m\sigma \Big|_0^\pi = \frac{-1}{m}(\cancelto{0}{\sin m\pi} - \cancelto{0}{\sin 0}) = 0 \,. \tag{14}$$

2 Harmonic Oscillations

For the generic variable x, KE $= \dfrac{m}{2}\dot{x}^2$, choose mass so that $m = 1$. KE $= \dfrac{1}{2}\dot{x}^2$. PE $= \frac{\kappa}{2}x^2$. Now, let $\omega \equiv \sqrt{\kappa/m} = \sqrt{\kappa}$. Then,

$$E = \frac{\dot{x}^2}{2} + \frac{\omega^2}{2} \,. \tag{15}$$

And the Lagrangian taking the form:

$$\mathscr{L} = \frac{\dot{x}^2}{2} - \frac{\omega^2}{2} \,. \tag{16}$$

3 Properties of 'particles'

What do we mean by a 'particle'? At some scale, a particle has locality, yet it is never a point – not even the electron due to it having virtual photons around it. A particle cannot have an energy spectrum other than its rest energy (mass). For a particle, its energy states are well separated compared to its ground state energy. For the strings we are concerned with, then next excited state corresponds to the addition of the Planck mass, which is 2.176 $\times 10^{-8}$ Kg.

4 The Light-Cone Frame

We are interested in formulating a means to describe the string in non-relativistic terms. The trick is to boost the string in some arbitrary direction to near the speed of light, which will have the effect of, first, Lorentz contracting the string onto the plane perpendicular to the direction of motion, and, second, will slow down the motion of the string parts relative to their center of mass, due to Lorentz time dilation.

The center-of-mass frame tis that frame in which the total momentum of the collection (of whatever) is zero. We will call the mass of that system in the senter-of-mass frame as M. Then the energy is given as ($c = 1$):

$$E = \sqrt{P^2 + M^2} = \sqrt{p_z^2 + p_x^2 + p_y^2 + M^2} \,, \tag{17}$$

where the z direction is the boost direction. If we boost enough in the z direction, we can make $|\, p_z \,| >> |\, p_x \,|, |\, p_y \,|$

$$E \sim p_z + \frac{p_x^2 + p_y^2}{2p_z} + \frac{M^2}{2p_z} \,. \tag{18}$$

So, what about the very large p_z term? This is the magic. p_z has become essentially a constant of the motion, which means that it can be subtracted from the equation.

Energy is related to time

$$H = i\frac{\partial}{\partial t} \,. \tag{19}$$

Since

$$\frac{p_x^2 + p_y^2}{2p_z} + \frac{M^2}{2p_z} \longrightarrow 0\,, \tag{20}$$

then $i\dfrac{\partial}{\partial t} \to 0$. So that the waveform changes slowly. Even though we went to all this effort to slow the system down, we need to rescale time to speed this up so we can analyze the time evolution of the x, y components. But how? Throw out p_z! (Ha! Ha!)

$$\text{New Hamiltonian} = (E - p_z)p_z = \underbrace{\frac{p_x^2 + p_y^2}{2}}_{\text{CM motion}} + \frac{M^2}{2}\,. \tag{21}$$

5 Strings moving in two dimensions

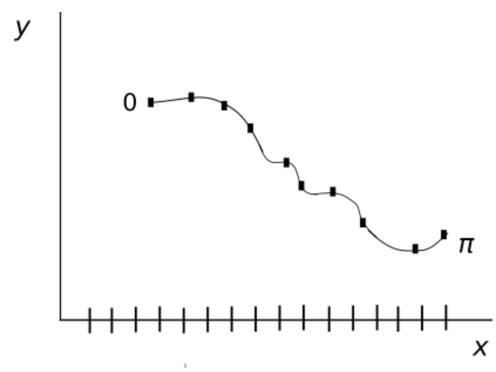

Figure 3. A relativistic string moving in 2-d, modeled as a collection of N connected mass points, with $N-1$ spring-like forces between them.

The non-relativistic energy of the string model above is given as

$$E = \sum \frac{\mu \dot{x}_i^2}{2} + \sum \frac{M\dot{y}_i^2}{2} + \frac{\kappa \Delta x_i^2}{2}\,, \tag{22}$$

where κ is the spring constant. We need a parameter to locate the mass points on this curve, which we'll call σ. But we have to adjust the mass of the points to maintain the overall mass of the string as M.

$$\mu = \frac{1}{N}\,. \tag{23}$$

The spring constant K of the entire string goes as $1/N$:

$$K = \frac{N}{\pi^2}\,, \tag{24}$$

where the π^2 is used to make later equations simpler.

$$\mu = \frac{1}{N}\,. \tag{25}$$

$$\Delta\sigma = \frac{\pi}{N}\,. \tag{26}$$

Then for the energy:

$$\begin{aligned}\text{KE} &\sim \sum \frac{1}{N}\frac{\dot{x}_i^2}{2} = \frac{1}{2\pi}\sum \Delta\sigma x_i^2 \\ &\to \frac{1}{2\pi}\int_0^\pi \left(\frac{\partial x}{\partial \tau}\right)^2 d\sigma\,,\end{aligned} \tag{27}$$

where τ is proper time. Similarly,

$$\text{PE} \sim \frac{1}{2\pi}\int_0^\pi \left(\frac{\partial x}{\partial \sigma}\right)^2 d\sigma\,. \tag{28}$$

Hence,

$$E = \frac{1}{2\pi}\int_0^\pi \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial x}{\partial \sigma}\right)^2 d\sigma\,, \tag{29}$$

which takes the form of the energy of a wave field. If a wave travels along the string, it will reflect (flipped over under Dirichlet BC) to return the other way. Under Neumann BC, it reflect as it entered the boundary point. And

$$\mathscr{L} = \frac{1}{2\pi}\int_0^\pi \left(\frac{\partial x}{\partial \tau}\right)^2 - \left(\frac{\partial x}{\partial \sigma}\right)^2 d\sigma\,, \tag{30}$$

The force on the Nth particle goes as $N\dfrac{\partial x}{\partial \sigma}\Delta\sigma$.

$$\frac{\partial x}{\partial \sigma} \propto F = \mu\ddot{x} = \frac{1}{N}\ddot{x}\,. \tag{31}$$

Therefore,

$$\ddot{x} \propto N\frac{\partial x}{\partial \sigma} \tag{32}$$

at an end point. This means that the acceleration goes to infinity. To avoid this, we impose the Neumann B.C. (zero at the end point). Then,

$$\frac{\partial x}{\partial \sigma} \equiv 0\,. \tag{33}$$

So, at this point in the analysis, we have a Lagrangian and adequate boundary conditions to work with.

Next, we compute the quantum mechanical energy levels, which will tell us something about the masses of these objects. Let's Fourier expand x and y:

$$x(\sigma,\tau) = \sum_{n=0}^\infty x_n \cos n\sigma\,, \tag{34}$$

$$y(\sigma,\tau) = \sum_{n=0}^\infty y_n \cos n\sigma\,. \tag{35}$$

with x_n, y_n as new degrees of freedom.

$$\frac{\partial x}{\partial \tau} = \dot{x}(\sigma) = \int_0^\pi \sum_{n=0}^{\infty} \dot{x}_n \cos n\sigma \,, \tag{36}$$

$$\left(\frac{\partial x}{\partial \tau}\right)^2 = \frac{1}{2\pi} \sum_{\substack{n=0\\m=0}}^{\infty} \int_0^\pi (\dot{x}_n \cos n\sigma)(\dot{x}_m \cos m\sigma) d\sigma \tag{37}$$

$$= \frac{1}{2\pi} \sum \dot{x}_n \dot{x}_m \int_0^\pi \cos n\sigma \cos m\sigma d\sigma \,. \tag{38}$$

$$\text{KE} = \frac{1}{2\pi} \sum_{\substack{n=0\\m=0}}^{\infty} \sum \dot{x}_m \dot{x}_n \int_0^\pi \cos n\sigma \cos m\sigma \,. \tag{39}$$

But we have the simplifying relation:

$$\int_0^\pi \cos m\sigma \cos n\sigma \, d\sigma = \delta_{nm} \frac{\pi}{2} \,, \tag{40}$$

when not both $n = 0$ and $m = 0$. Fortunately, we can split this into parts:

$$\text{KE} = \frac{\dot{X}_0^2}{2} + \frac{1}{2\pi} \sum \dot{x}_n^2 (\pi/2) = \frac{\dot{X}_0^2}{2} + \frac{1}{4} \sum \dot{x}_n^2 \tag{41}$$

where X_0 is the center-of-mass position.

$$\frac{\partial x}{\partial \sigma} = -\sum_{n=0}^{\infty} n \dot{x}_n \sin n\sigma \,. \tag{42}$$

$$\text{PE} = \left(\frac{\partial x}{\partial \sigma}\right)^2 = \frac{1}{2\pi} \sum_{\substack{n=0\\m=0}}^{\infty} nm \int_0^\pi x_n x_m \sin m\sigma \sin n\sigma \, d\sigma \tag{43}$$

Now, we have that

$$\int_0^\pi \sin m\sigma \sin n\sigma \, d\sigma = \delta_{nm} \frac{\pi}{2} \,. \tag{44}$$

Thus, perfoming the integration on (43), we get that

$$\text{PE} = \tfrac{1}{4} \sum_{n}^{\infty} n^2 x_n^2 \,. \tag{45}$$

Back to the Lagrangian:

$$\mathscr{L} = \frac{\dot{X}_0^2}{2} + \tfrac{1}{4} \sum_{n}^{\infty} n^2 \dot{x}_n^2 - \tfrac{1}{4} \sum n^2 x_n^2 \,. \tag{46}$$

Thus, for each n, there is an nth harmonic oscillation

$$\mathscr{L}_n = \tfrac{1}{4} n^2 \dot{x}_n^2 - \tfrac{1}{4} n^2 X_n^2 \qquad \text{(where } \omega_n = n \text{ is the frequency)} \,. \tag{47}$$

And now we will drop the CM term!

$$\mathscr{L} = \tfrac{1}{4} \sum_n \dot{x}_0^2 - \tfrac{1}{4} \sum_n n^2 x_n^2 \,. \tag{48}$$

Going back to the energy, we have

$$E = \tfrac{1}{4}\sum_n \dot{x}_0^2 + \tfrac{1}{4}\sum_n n^2 x_n^2\,, \tag{49}$$

which is the internal energy; and this must be identified with the mass squared of the particle. However, note that we have only performed our calculation in the x direction, thus, we must add in similar terms for the y direction.

Next, we quantize these modes, which means to quantize the energies.

All this is duplicated in the y-axis. In the ground state, where there are no oscillation, we'll refer to it as $|\,0\,\rangle$.

$$E = \hbar q\omega\,, \tag{50}$$

where q is another integer that labels the number of quanta, and we will set $\hbar$ to unity and $\omega = n$.

$$E_n = qn\,. \tag{51}$$

For the first excited state, we need $q = n = 1$, so we can excite either the x or the y:

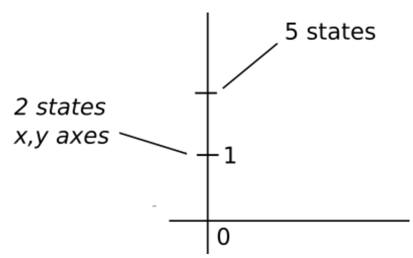

Figure 4. Excited states, taken relative to an unspecified ground state.

The next excited state can be x, x or y, y or the lowest x and the lowest y, or excited the 2nd oscillation once: $\dot{x}_2$ or $\dot{y}_2$.

Let a for x and b for y:

$a_1^{\dagger\,2}\,|\,0\,\rangle$ – x osc.

$b_1^{\dagger\,2}\,|\,0\,\rangle$ – y osc.

$a_1^{\dagger}b_1^{\dagger}\,|\,0\,\rangle$ – y osc.

$a_2^{\dagger}\,|\,0\,\rangle$ – x.

$b_2^{\dagger}\,|\,0\,\rangle$ – y.

Note: Open strings behave as photons; closed strings as gravitons.