String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 3

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Abstract

This paper contains my notes on Lecture Three of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belong solely to me.

"Negative mass squared is a bad thing." − Susskind

1 Preliminaries

The harmonic ocsillator energies of the nth oscillation:

$$E_n = \tfrac{1}{4}\dot{x}_n^2 + \tfrac{1}{4}n^2x_n^2\,, \tag{1}$$

and Lagrangian

$$\mathscr{L} = \tfrac{1}{4}\dot{x}^2 - \tfrac{1}{4}n^2x^2\,. \tag{2}$$

Next, to develop the Hamiltonian,[1] we first replace $\dot{x}$ by momentum

$$p = \frac{\dot{x}}{2}\,, \tag{3}$$

where the division by 2 comes from the 1/4th in the previous equation. Next, we need E as a function of x and p:

$$E(p,x) = p^2 + \tfrac{1}{4}n^2x_n^2 = H\,. \tag{4}$$

Then, we factor H

$$H = \big(\frac{nx}{2} + ip\big)\big(\frac{nx}{2} - ip\big)\,. \tag{5}$$

And for the raising and lowering operators, we have that

$$[\,a^-, a^+\,] = 1\,. \tag{6}$$

Note: $[\,x,p\,] = i$, where we have set $\hbar = 1$.

$$[\,\big(\frac{nx}{2} + ip\big), \big(\frac{nx}{2} - ip\big)\,] = \frac{n}{2}(-i)[\,x,p\,] + \frac{n}{2}(-i)i = n\,. \tag{7}$$

[1]Finding the Hamiltonian is the first step in the process of transforming to the quantum mechanical situation.

$$a^- = \frac{\sqrt{n}x_n}{2} + \frac{i}{\sqrt{n}p_n}\,, \tag{8a}$$

$$a^+ = \frac{\sqrt{n}x_n}{2} - \frac{i}{\sqrt{n}p_n}\,, \tag{8b}$$

where a^- and a^+ are complex conjugates of each other. Of course, the full Hamiltonian also needs the y terms. We'll reserve a for the x axis and b for the y axis.

$$[\,b^-, b^+\,] = 1\,. \tag{9}$$

Solve for x_n from (8a) and (8b).

$$a^+ + a^- = \sqrt{n}x_n\,. \tag{10}$$

So, for x_n as a function of the annihilation-creation operators, we get that

$$x_n = \frac{a^+ + a^-}{\sqrt{n}}\,, \quad y_n = \frac{b^+ + b^-}{\sqrt{n}}\,, \tag{11}$$

where we have thrown in the corresponding result for y_n.

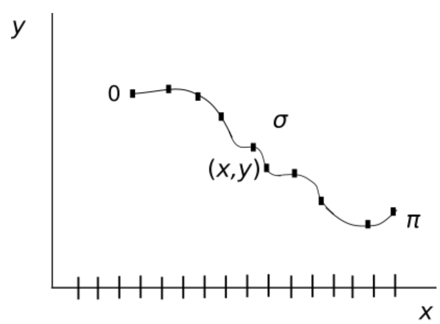

Figure 1. A string in the x, y-plane, modeled as a collection of N mass points, with $N-1$ spring-like forces between them. The 'position' of the points on the string is parametarized by σ. The reason to start at zero and end at π is to model an open string.

$$x(\sigma) = \sum_n x_n \cos n\sigma\,, \qquad y(\sigma) = \sum_n y_n \cos n\sigma\,, \tag{12}$$

where x_n and y_n are the Fourier coefficients of the cosine waves, and they are functions of time τ, not of σ. We need the net force on the end points to be zero, which forces the relation

$$\dot{x}(\sigma) \sim \sin n\sigma \to 0\,. \tag{13}$$

So that,

$$x(\sigma) = \sum_n \frac{a_n^+ + a_n^-}{\sqrt{n}} \cos n\sigma\,, \tag{14a}$$

$$y(\sigma) = \sum_n \frac{b_n^+ + b_n^-}{\sqrt{n}} \sin n\sigma\,. \tag{14b}$$

2 Spin

For a particle moving along the x direction, we use the right-hand rule convention to make the direction of the spin along this axis.

For spin 1massive (has mass) particles, there are three possibilities along the axis: $-1, 0, +1$.

If spin of a massive starts off $+1$, you can boost to its rest-mass frame and then rotate the particle so that its spin direction has no component along the z axis: Spin along the z-axis is then zero. Then you just re-boost the particle to its previous speed, and voila, no spin in the z direction!

However, if the particle is massless, then there's no way to change its spin value and no spin zero state.

Linear polarization of a photon is a quantum superposition of photons equally $+1$ spin and -1 spin.

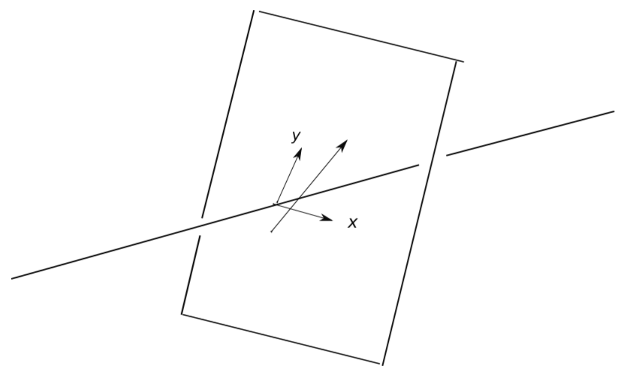

Figure 2. Photon Polarization is perpendicular to the direction of motion (transverse).

Two states are allowable: $|\,x\,\rangle$, $|\,y\,\rangle$.

What about circular polarization? For this we need a superposition of $|\,x\,\rangle$ and $|\,y\,\rangle$ to describe right polarization $|\,r\,\rangle$ and left polarization $|\,\ell\,\rangle$.

$$|\,r\,\rangle = |\,x\,\rangle + i\,|\,y\,\rangle \;, \tag{15a}$$
$$|\,\ell\,\rangle = |\,x\,\rangle - i\,|\,y\,\rangle \;. \tag{15b}$$

This is a circular polarization per photon, allowing for just two states, and the same goes for the graviton. By the way, a graviton with two units of angular momentum in the z direction is mathematically equivalent to a photon pair with their angular momenta adding up in the z direction.

3 Energy levels of a vibrating string

$E \to m^2$ [total energy of the vibrating string]. What are the discrete energy levels of a string (at rest)?[2] It's a collection of harmonic oscillations. The lowest state (the ground state $|\,0\,\rangle$) is

[2] The 'low-lying spectrum' is the first few states.

annihilated by all the annihilation operators:

$$a_n^- \,|\,0\,\rangle \,= 0\,, \tag{16a}$$
$$b_n^- \,|\,0\,\rangle \,= 0\,, \tag{16b}$$

where $|\,0\,\rangle$ is the unexcited oscillators (not the vacuum) and the n's include all of them.

We'll indicate the energy of the ground state as (it won't be $\frac{1}{2}\hbar$)

$$E_{\text{GS}} = m_0^2\,, \tag{17}$$

and we don't yet know what m_0 is. The oscillation of smallest frequency is a_1^+,

$$a_1^+\,|\,0\,\rangle \,= m_0^2 + 1\,, \tag{18}$$

$E = m_0^2 + 1$. And b_1^+ has $m_0^2 + 1$.

$$x = \sum a\,, \tag{19a}$$
$$y = \sum b\,, \tag{19b}$$

where a and b are components of a vector. Since we have a rotation symmetry in the x, y plane, under a rotation of coordinates about the z axis, a_1^+ and b_1^+ are vectorial in the x, y plane. To get circular polarization, we take $(a_1 \pm ib_1)\,|\,0\,\rangle$.

Since there are only two polarizations states, $a_1^+\,|\,0\,\rangle$ and $b_1^+\,|\,0\,\rangle$ act like photon-like states (massless). Then, we need to set $m_0 + 1 = 0$ implies

$$m_c^2 = -1\,. \tag{20}$$

For a nonrelativistic particle

$$E(p) = \frac{p^2}{2m}\,, \tag{21}$$
$$\frac{\partial E}{\partial p} = \frac{p}{m} = v \quad \text{or}\,, \tag{22}$$
$$\frac{\partial \omega}{\partial k} = v\,. \tag{23}$$

For a photon,

$$E = cp\,, \tag{24}$$
$$\frac{\partial E}{\partial p} = c\,. \tag{25}$$

Now,

$$E = \sqrt{p^2 + m^2}\,, \tag{26}$$
$$\frac{\partial E}{\partial p} = \frac{p}{\sqrt{p^2 + m^2}} < 1\,. \tag{27}$$

(So take that, Tachyon!) But if $E = \sqrt{p^2 - m^2}$ then

$$\frac{\partial E}{\partial p} = \frac{p}{2\sqrt{p^2 - m^2}} > 1 \tag{28}$$

for tachyons. However, their existence would mean that spacetime is unstable.

Now, the wave equation is

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}\,, \tag{29}$$

with

$$k^2 = \omega^2 \tag{30}$$

with speed

$$\frac{d\omega}{dk} = 1\,. \tag{31}$$

Add a mass term to get a speed less than light.

$$-m^2\phi + \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}\,, \tag{32}$$

with (on substituting in for ϕ and simplifying)

$$-m^2 - k^2 = -\omega^2 \tag{33}$$

or

$$\omega^2 = k^2 + m^2\,. \tag{34}$$

The energy of the wave is

$$\tfrac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \tfrac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{m^2\phi}{2}\,, \tag{35}$$

where the last term is the potential energy for the field. For a negative sign in front of the PE term we have instability. Nearby states will destablize ($-m^2$ scenario), but the speed of propagation is less than c. To solve this problem we need **superstring theory**. Open strings can rotate about its CM, but this requires energy and angular momentum.

Note: Any continuous function on 0 to π can be expanded in sines and cosines. This brings us to the important issue of boundary conditions (BC). We have two of specific interest:

Dirichlet B.C. or Neumann B.C. For Dirichlet, we have $x(0) = 0$ and $x(\pi) = 0$. For Neumann BC, we have $\frac{\partial x(0)}{\partial \sigma} = 0$ and $\frac{\partial x(\pi)}{\partial \sigma} = 0$.

The Fourier expansion of the Dirichlet BC:

$$x(\sigma) = \sum_{n=1}^{\infty} x_n \sin n\sigma\,. \tag{36}$$

The sine is zero at $\sigma = 0$, $\sigma = \pi$

Neumann B.C.

$$x(\sigma) = \sum_{n=0}^{\infty} x_n \cos n\sigma\,, \tag{37}$$

$$\dot{x}(\sigma) = 0 \text{ at } \sigma = 0\,,\ \sigma = \pi\,. \tag{38}$$

4 String Coupling Constant

Two open strings can join at their ends.

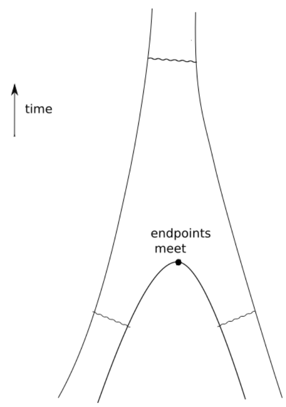

Figure 3. Two open strings can meet and join at their endpoints to make a new string. And, because this is quantum mechanics, this process is reversible.

The life expectancy of a combined string depends on the string coupling constant between any two strings. The coupling constant measures the probability of a chance of status. If it is small, the stability of the string is large. If it is large, the combined particle might survive a short time.

For example, the probability of an interaction of an electron and a photon is the square of the fine structure constant (about α^2), which goes as the electric charge.

It's possible to have a theory of closed strings, but not a theory of only open strings. All string theories have closed strings – which must contain gravity. In string theory, every particle can absorb a graviton, which is why all mass has gravity.