String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 4

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Abstract

This paper contains my notes on Lecture Four of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belong solely to me.

1 Preliminaries

Suppose we start with the Lagrangian $\mathscr{L}(q, \dot{q})$. Then

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \,. \tag{1}$$

and Lagrangian

$$\mathscr{L} = \tfrac{1}{4}\dot{x}^2 - \tfrac{1}{4}n^2 x^2 \,. \tag{2}$$

Next, we develop Noether's Theorem. Now, suppose we have an infinitesimal symmetry, given by

$$\delta q_i = f_i(q)\epsilon \,. \tag{3}$$

Then the Noether Charge ("generator" in QM) is given as

$$Q_N = \sum_i p_i f_i(q)\epsilon \,, \tag{4}$$

and is a conserved quantity.

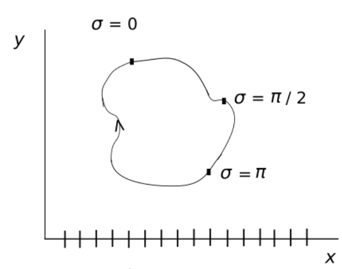

Figure 1. A closed string in the x, y-plane, modeled as a collection of N mass points, and parameterized by σ, which starts at 0 and goes around the loop. until it reaches the start point, now $\sigma = 2\pi$.

With reference to Fig. 1, imagine lifting the string out of the xy plane and flipping it and then setting it back into the xy plane. By doing this, we have changed the orientation of the string, but not of the plane.

2 A wave along the string

Definition: If a wave travels along the string in the direction pof increasing σ, we'll refer to that as the 'right' direction. Going in the opposite direction is in the 'left' direction.

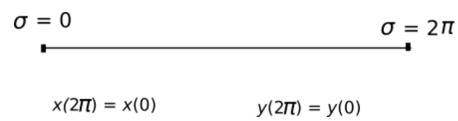

Figure 2. The σ axis, one for the x direction, one for the y direction.

We will represent waves by exponentiiation:

$$e^{in\sigma} = \cos n\sigma + i \sin n\sigma \,. \tag{5}$$

For a periodic wave,

$$x(\sigma) = \sum_n x_n e^{in\sigma} \,. \tag{6}$$

For right-moving waves:

$$x_R(\sigma) = \sum_{n>0} x_n e^{in\sigma} \,. \tag{7}$$

For left-moving waves

$$x_L(\sigma) = \sum_{n>0} x_{-n} e^{-in\sigma} \,. \tag{8}$$

$$x(\sigma) = x_R + x_L + x_0 \,, \tag{9}$$

or

$$x(\sigma) = \sum_{n>0} x_n e^{in\sigma} + \sum_{n>0} x_{-n} e^{-in\sigma} + x_0 \,, \tag{10}$$

where x_0 is the center of mass of the string. Of course, a similar equation applies to the y coordinate.

3 Lagrangian and Energy

$$\mathscr{L} = \int_0^{2\pi} d\sigma \left[\left(\frac{\partial x}{\partial \tau} \right)^2 - \left(\frac{\partial x}{\partial \sigma} \right)^2 \right] = \text{a } y \text{ term} \,. \tag{11}$$

This yields form the equation of motion harmonic oscillations.

Energy:

$$E = \int_0^{2\pi} d\sigma \left[\left(\frac{\partial x}{\partial \tau} \right)^2 + \left(\frac{\partial x}{\partial \sigma} \right)^2 \right] \,, \tag{12}$$

where $\left(\frac{\partial x}{\partial \tau} \right)^2$ is the energy of the left-moving wave, and $\left(\frac{\partial x}{\partial \sigma} \right)^2$ is the energy of the right-moving wave.

Harmonic oscillations: n represents frequency.

Creation operators for x:

$$a_n^+ \,, \quad a_{-n}^+ \,. \tag{13}$$

Creation operators for y:

$$b_n^+ \,, \quad b_{-n}^+ \,. \tag{14}$$

We take the ground satate as $|\,0\,\rangle$. To excite this state to the very next energy level, we use

$$a_1^+ \,|\,0\,\rangle \quad \text{and} \quad b_1^+ \,|\,0\,\rangle \ . \tag{15}$$

(For a photon, these correspond to polarization.)

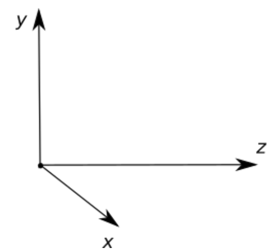

Figure 3. Consider the string to move along the z direction.

The string is traversing in the z direction with a huge momentum, but its oscillations are along the x and y axes.

For circular polarization, we form the construct

$$(a_1^+ \pm b_1^+)\,|\,0\,\rangle \ . \tag{16}$$

We start with the ground state with energy m_0^2. Thew lowest excitations on $|\,0\,\rangle$ is accomplished by operators $a_1^+\,, a_{-1}^+\,, b_1^+\,, b_{-1}^+$, making four distinct states, or

$$a_{-1,1}^+, \quad b_{-1,1}^+ \quad \text{on} \quad |\,0\,\rangle \ . \tag{17}$$

However, this is not the right approach. $a \pm ib$ won't give a state of angular momentum zero. Hence, these photons are massless. But if we go to the next level of angular momentum, we won't get correct results.

4 The Rule of Level Matching

The right and left-moving energies must be the same, which will be explained later. So, what state can we have? Does $a_1^+\,|\,0\,\rangle$ have one unit of right-moving energy? Nope. Does $a_{-1}^+\,|\,0\,\rangle$ have one unit of left-moving energy? Nope. Etc.

At the next level, we get two units of energy, as much left as right:

$$a_1^- a_{-1}^+\,|\,0\,\rangle = \text{Ok for a state!}\,, \tag{18a}$$

$$b_1^- b_{-1}^+\,|\,0\,\rangle = \text{Ok for a state!}\,, \tag{18b}$$

$$a_1^- b_{-1}^+\,|\,0\,\rangle = \text{Ok for a state!}\,, \tag{18c}$$

$$a_{-1}^+ b_1^-\,|\,0\,\rangle = \text{Ok for a state!}\,. \tag{18d}$$

Alternatively, a_2^+ , a_{-2}^+ , b_2^+ , b_{-2}^+ each provide two units of energy, but they do not satisfy the Principle of Level Matching.

We also have circular polarization to consider. We can also form circularly polarized operators with two units of angular momentum:

$$(a_1^+ ib_1^+)(a_1^+ ib_1^+)\,|\,0\,\rangle\ . \tag{19}$$

But for level matching,

$$(a_1^+ + ib_1^+)(a_{-1}^+ + ib_{-1}^+)\,|\,0\,\rangle \qquad (m=2)\,. \tag{20}$$

where the left operator is right-moving, and the right operator is left-moving. Also,

$$(a_1^+ - ib_1^+)(a_{-1}^+ - ib_{-1}^+)\,|\,0\,\rangle \qquad (m=-2)\,, \tag{21}$$

gives us two left-handed photons with angular momentum -2. Next,

$$(a_1^+ + ib_1^+)(a_{-1}^+ - ib_{-1}^+)\,|\,0\,\rangle \qquad (m=0)\,, \tag{22}$$

with total angular momentum zero. And, finally,

$$(a_1^+ - ib_1^+)(a_{-1}^+ + ib_{-1}^+)\,|\,0\,\rangle \qquad (m=0)\,, \tag{23}$$

Unlike the case of a photon (which is massless), a massive spin-2 particle has five states, given by $2, 1, 0, -1, -2$. However, we don't have particles that correspond to all these states. As a consequence, suppose we can identify this particle as a spin-2 graviton. But what should we make of the two particles with zero angular momentum?

Dilaton or axion are particles corresponding to zero mass. These are familiar particle that arise in string theory, but they have never been observed.

Desideratum: To get rid of the $m=0$ particles but keep the $m=2$ particles.

Is there a special spot on the string to place the point $\sigma=0$? If not, then the theory must be invariant under a shift in the parameter σ. For starters, we think of our closed string as a descrete collection of points, labeled from $1 \ldots N$. Ignoring the y coordinates for now, the quantum state vector is

$$\psi(x_1, x_2, \ldots, x_N) \quad \text{or} \quad \psi(x_2, \ldots, x_N, x_1)\,. \tag{24}$$

Clearly, we want ψ to be invariant under cyclic permutation of the index numbering.

Now, we switch to a continuous variable and express analogous idea:

$$\psi(x(\sigma)) = \psi(x(\sigma+\epsilon))\,, \tag{25}$$

or, re-expressed as

$$\psi(x(\sigma)) - \psi(x(\sigma+\epsilon)) = 0\,, \tag{26}$$

which is a differential quantity that can be expressed as

$$\frac{\partial\psi}{\partial(x(\sigma))}\frac{\partial x}{\partial\sigma}\epsilon = 0 \tag{27}$$

at each point. On dropping the ϵ and integrating, we have that

$$\int \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\sigma}\,d\sigma = 0\,. \tag{28}$$

In quantum mechanics, the variable p is treated as the following operator:

$$p = -i\frac{\partial}{\partial q}\,. \tag{29}$$

Therefore,

$$\frac{\partial \psi}{\partial q} = ip\psi\,, \tag{30}$$

and (28) becomes

$$\int p(\sigma)\frac{\partial x}{\partial \sigma}\,d\sigma = 0\,. \tag{31}$$

Each point on the string has its own momentum $p(\sigma)$. But

$$m\frac{\partial x}{\partial \sigma} = p\,. \tag{32}$$

Therefore,

$$\int \dot{x}(\sigma)\frac{\partial x}{\partial \sigma}\,d\sigma = \int \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \sigma}\,d\sigma = 0\,, \tag{33}$$

to maintain that there is no preferred point on the sigma coordinate.

$$\int_0^{2\pi} \tfrac{1}{2}\,d\sigma \left[\left(\frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial \sigma}\right)^2 - \tfrac{1}{2}\left(\frac{\partial x}{\partial \tau} - \frac{\partial x}{\partial \sigma}\right)^2\right] = 2\int \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \sigma}\,d\sigma\,. \tag{34}$$

Thus,

$$2\int \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \sigma}\,d\sigma = 0 \tag{35}$$

implies that the left-moving energy and the right-moving energy are the same. Thus, the $m = 0$ states are eliminated. There are still higher-level possibilities, though we won't go into them.

Big Question: Are strings the ultimate building blocks of particle physics?

To answer this question, we'd have to go into the realm of QED.

A Comparative Question: Which is more fundamental: The electron or the monopole?

Now, the electric charge e and the monopole charge q must satisfy the following constraint:

$$e \cdot q = 2\pi \tag{36}$$

in order for the Dirac string (which is the solenoid) to be invisible.

A Feynmann diagram analysis requires charge to be small to have convergence.

Okay, back to $e \cdot q = 2\pi$. If e is small then q must be big. Now, since Maxwell's equations are nearly symmetric under interchange of electric and magnetic fields, we should be able to swap e and q. Does this help us determine which of the two is the more fundamental? The answer requires us to return to Feynmann diagrams, in which we find that this particular swap because we have serious lack of convergenses. Corresponding to the higher charge of the monopole, we expect it to have more field energy and more mass. Therefore a mgnetic charge should be effective in emitting a photon and virtual photons, making the monopole a complicated mess to deal with. Too complicated, in fact, even for a Feynmann diagram.