String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 6

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Abstract

This paper contains my notes on Lecture Six of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belong solely to me.

1 Scattering Experiments

We begin by setting $c = \hbar = 1$.

Scattering experiments are central to particle physics because they are at the heart of what physicists can do to prove the predictions of their theories.

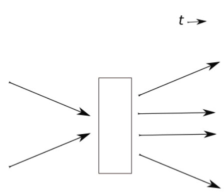

Figure 1. These particles have 4-momentum.

We analyze the interaction through conservation of 4-momenta.

$$k_\mu = (E, p_x, p_y, p_z) \quad (\mu = 0, 1, 2, 3)\,, \tag{1}$$

and equations to obey

$$E^2 = \mathbf{p}^2 + m^2 \quad \text{or} \quad E^2 - \mathbf{p}^2 = m^2\,. \tag{2}$$

Or even

$$\mathbf{p}^2 - E^2 = -m^2 \quad \text{or} \quad \mathbf{k}^2 - k_0^2 = -m^2\,. \tag{3}$$

Or more succinctly

$$p_\mu p^\mu = k^2 = -m^2\,. \tag{4}$$

We're taking k^2 to be the same for every particle. The 4-momentum equation:

$$k_1 + k_2 = q_3 + q_4\,. \tag{5}$$

Let $q_3 \to -k_3$ and $q_4 \to -k_4$.

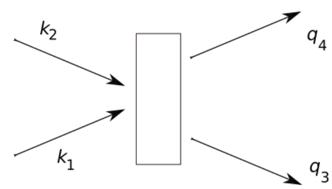

Figure 2. Two particles in; two particles out.

Hence,

$$k_1 + k_2 + k_3 + k_4 = 0\,, \tag{6}$$

subject to the constraint $k_i^2 = -m^2$ for each i.

Question: What are the amplitudes for this collision and results? $A(k_1, k_2, k_3, k_4)$.

Now, we are allowed to move to the center of mass frame in which the CM is at rest. We can also orient the axes so that our motion is along the x-axis. In the CMF the only thing to analyze is the total energy. For simplicity, assume all the particles are of the same kind.

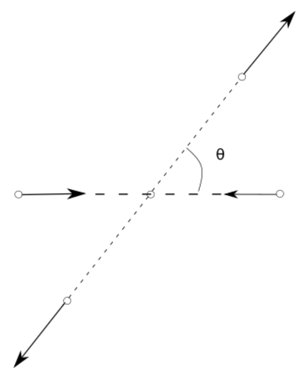

Figure 3. Collision as seen in the CMF: all particles are the same.

We are down to two independent variables. We should be able to characterize this simple scattering with only two relativistically invariant parameters.

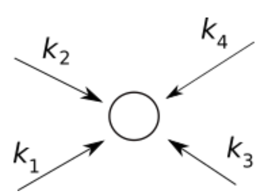

Figure 4. A further simplification. All particles have the same energy.

How to produce an invariant?

$$(k_1 + k_2)^2 \equiv (\mathbf{k}_1 + \mathbf{k}_2)^2 - (k_{01} + k_{02})^2\,, \tag{7}$$

where $(\mathbf{k}_1 + \mathbf{k}_2)^2$ is zero in the CM frame and $(k_{01} + k_{02})^2 = (2k_0)^2$ is the CM energy. Let S be the CM energy.

$$S = (k_1 + k_2)^2 = (\mathbf{k}_1 + \mathbf{k}_2)^2 = E_{\mathrm{CM}}^2\,, \tag{8}$$

$$(k_1 + k_3)^2 = (\mathbf{k}_1 + \mathbf{k}_3)^2 - (k_{01} + k_{03})^2 = (\mathbf{k}_1 + \mathbf{k}_3)^2 \,. \tag{9}$$

But $k_{01} + k_{03} = 0$ because they have opposite signs.

$$s = E_{\mathrm{CM}}^2 \,, \tag{10}$$

Now,

$$t \equiv (E_{\mathrm{CM}}^2 - m^2)(1 - \cos\theta) \,, \tag{11}$$

$$-t = (k_1 + k_3)^2 \,, \tag{12}$$

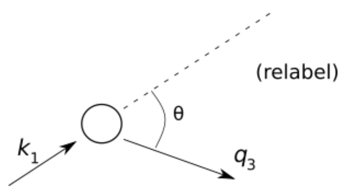

Figure 5. A momentum transfer from particle 1 to 3.

$$(\mathbf{k}_1 + \mathbf{k}_3)^2 = (\mathbf{k}_1 - \mathbf{q}_3)^2 \quad \text{(momentum transfer)} \,. \tag{13}$$

Then

$$(k_1 + k_3)^2 = 2(E^2 - m^2)(1 - \cos\theta) \,. \tag{14}$$

So

$$-u = (k_1 + k_4)^2 \,, \tag{15}$$

which is not independent of the other two.

The Madelstam Variables (from Wikipedia):

$$s = (p_1 + p_2)^2 c^2 = (p_3 + p_4)^2 c^2 \,, \tag{16}$$
$$t = (p_1 - p_3)^2 c^2 = (p_4 - p_2)^2 c^2 \,, \tag{17}$$
$$u = (p_1 - p_4)^2 c^2 = (p_3 - p_2)^2 c^2 \,, \tag{18}$$

Make conversion of variables as needed.

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2 Feynman Diagrams

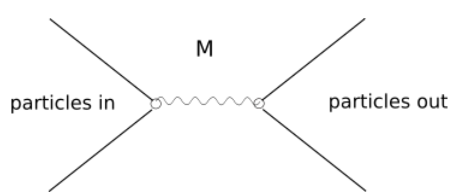

Figure 6. An s-channel process. g is the coupling constant.

The characteristic structure of the scattering amplitude is

$$g^2 \frac{1}{s - M^2} \,, \tag{19}$$

Alternatively, we can write down a t-channel process with amplitude $g^2 \dfrac{1}{t - m^2}$:

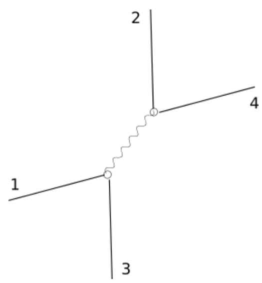

Figure 7. A t-channel process. g is the coupling constant.

Here, all scattering angles are equally probable. The Feynman diagram explanation isd the connection pipe between he incoming and outgoing particles is forgetful.

On combining the relevant Feynman diagrams, we get

$$g^2 \frac{1}{s - m^2} + g^2 \frac{1}{s - m^2} \,, \tag{20}$$

In the 1960s, the following formulation arose from trial and error: Gabriele Veneziano

Amplitude:

$$g^2 \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} \to s \,. \tag{21}$$

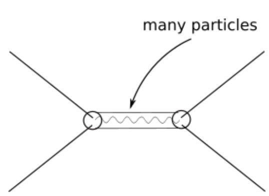

Figure 8a. An s-channel process.

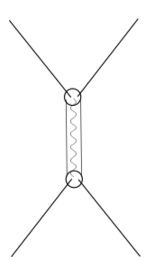

Figure 8b. A t-channel process.

Both figures above (Fig. 8a and 8b) represent the same math.

The theory that arose that best explained (21) is called *string theory.*

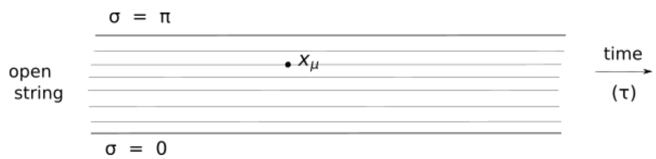

Figure 9. A The string sweeps out a world sheet.

The equation of motion of x are wave motion, describing waves moving up and down the string.

$$\frac{\partial^2 x}{\partial \tau^2} - \frac{\partial^2 x}{\partial \sigma^2} = 0\,. \tag{22}$$

So, we re-imaging the incoming particles as open strings. The collision of such particles is modeled by the strings meeting at end points as in the figure below.

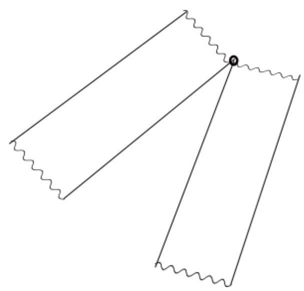

Figure 10. The strings meet at conjoining points to form a single new string.

We will overload τ this time to mean the time it takes between the formation of a single string and its breakup.

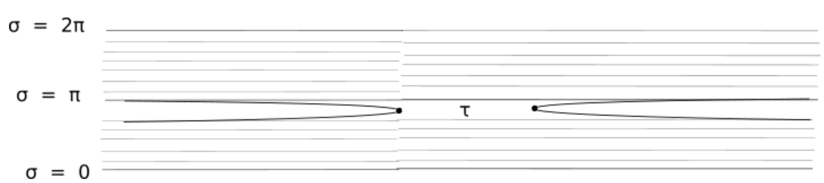

Figure 11. τ is the time between formation and breakupp..

To analyze this process quantum mechanically, we need an intial state: this will be the collection of mass points used to approximate the string, $x_1, \ldots, x_N$. Then we form the wave function $\psi(x_1, \ldots, x_N)$ at the start. So, ψ has momentum k.

$$\psi(x_1, \ldots, x_N) = e^{ik_1\left(\dfrac{x_1 + \cdots + x_N}{N}\right)}\psi_0(x_1, \ldots, x_N)\,, \tag{23}$$

where $\psi_0(x_1, \ldots, x_N)$ is the ground state. The CM position is $\dfrac{x_1 + \cdots + x_N}{N}$. Next, we need the wavefunction for the second particle:

$$\overline{\psi}(x_{N+1},\ldots,x_{2N}) = e^{ik_2\left(\dfrac{x_{N+1}+\cdots+x_{2N}}{N}\right)}\psi_0(x_{N+1},\ldots,x_{2N})\,, \tag{24}$$

and because $N >> 1$, we won't bother to change it in the denominator. However, to say that the two strings merge is to set x_{N+1} of the second string equal to x_N of the first string. Thus, the last equation becomes

$$\overline{\psi}(x_{N+1},\ldots,x_{2N}) = e^{ik_2\left(\dfrac{x_{N}+\cdots+x_{2N}}{N}\right)}\psi_0(x_{N+1},\ldots,x_{2N})\,. \tag{25}$$

Next, we evolve this new state by the Hamiltonian, and multiply by $e^{iH\tau}$, where H is the Hamiltonian for the $2N$ points of the combined system. The last step is to project the evolved system onto two new particles of momenta k_3 and k_4. Post separation we get

$$I = \int_0^\infty d\tau\, e^{\tau(s+1)}(1-e^{-\tau})^{-t-1}e^{-\tau}\,, \tag{26}$$

where $e^{\tau(s+1)}$ is the CM energy and t is the momentum change.

Lastly, we make the change of variable $e^{-\tau} \to z$:

$$I = \int_1^0 dz\, z^{-(s+1)}(1-z)^{-t-1}\,. \tag{27}$$

If we interchange z and $z-1$, we see that s and t are symmetric. This implies that the s-channel is convertible into the t-channel and vice versa.