String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 7

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Abstract

This paper contains my notes on Lecture Seven of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belong solely to me.

1 Scattering Experiments

Figure 1 shows the trajectory of a particle in classical relativity, having coordinates x^μ and classical action

$$s = \int dt \, \tfrac{1}{2}\dot{x}^2 \tag{1}$$

and relativstic action

$$s = \int d\tau \frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} \tag{2}$$

with τ being the proper time.

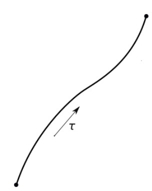

Figure 1. Trajectory of a particle in classical relativity.

However, in quantum mechanics we don't ask for trajectories; instead, we ask for probabilities.

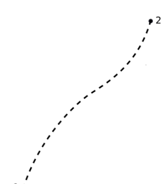

Figure 2. We need the amplitude of probability of transition from state 1 to state 2.

$$\mathrm{Amp}(1,2) = \sum_{\text{all traj.}} e^{\frac{i}{2}\int_{t_1}^{t_2} dt \left(\frac{dx}{d\tau}\right)^2} \to \int e^{\frac{i}{2}\int_{t_1}^{t_2} dt \left(\frac{dx}{d\tau}\right)^2} , \tag{3}$$

which is Feynman's path integral version of "least action." In relativistic physics, the amplitude (1,2) is called the *propagator*. For more complicated figures, like

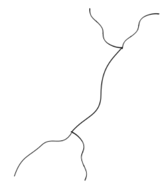

Figure 3. More complicated Feynman diagrams. Add together the actions for each segment. Integrate over the points where the trajectories come together. Lastly, integrate over all paths.

For more complicated Feynman diagrams, we add the individual segments together and sum/integrate over them. Then sum over all possible Feynman diagrams.

In the process of summing over all paths, the path far from the classical paths tend to concel out — and must be weakly weighed somehow.

Remark: All path described by the integral

$$e^{\frac{i}{2}\int_{t_1}^{t_2} dt \left(\frac{dx}{d\tau}\right)^2} \tag{4}$$

have equal "weigh" by virtue of the fact that

$$|\, e^{i\alpha} \,| = 1 \quad \text{for all } \alpha \,. \tag{5}$$

Anyway, we need some trick to get our integrals to converge. Let's return to the relativistic case $x^\mu(\tau)$:

$$\overline{s} = \int d\tau \left(\frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau}\right) , \tag{6}$$

where $\tau = \alpha\overline{s}$ and where α (not the same as the α just mentioned) is a parameter to be chosen later. Then

$$\overline{s} = \int_{s_1}^{s_2} \alpha d\overline{s} \left(\frac{\partial x}{\partial \overline{s}}\right)^2 \frac{1}{\alpha^2} = \int_{s_1}^{s_2} \frac{d\overline{s}}{\alpha} \left(\frac{\partial x^\mu}{\partial \overline{s}}\frac{\partial x_\mu}{\partial \overline{s}}\right) , \tag{7}$$

Therefore, we get

$$e^{\frac{i}{2}\int_{s_1}^{s_2} \frac{d\overline{s}}{\alpha} \left(\frac{\partial x^\mu}{\partial \overline{s}}\frac{\partial x_\mu}{\partial \overline{s}}\right)} = e^{\int_{s_1}^{s_2} \frac{i}{\alpha} d\overline{s} \left(\frac{\partial x^\mu}{\partial \overline{s}}\frac{\partial x_\mu}{\partial \overline{s}}\right)\frac{1}{2}} , \tag{8}$$

So, what happens on the paths that diverge far from the classical path or have large speeds? Then the $\left(\frac{\partial x^\mu}{\partial \overline{s}}\frac{\partial x_\mu}{\partial \overline{s}}\right)$ will be large and the exponential with the negative sign will go neglibly small. This new integral in known as the "Wiener Integral" of statistical mechanics.

There is a common trick in which time is replaced by imaginary time:

$$\int_{\text{all traj.}} e^{-\frac{1}{2}\int_{\overline{s}_1}^{\overline{s}_2} \left(\frac{\partial x}{\partial \overline{s}}\right)^2 d\overline{s}} = F(\overline{s}_1, \overline{s}_2) \,. \tag{9}$$

The next trick is to extrapolate values by analytic continuation into the complex plane.

2 String Theory

So, in string theory, a particle is modeled as a string. An open string is modeled as a ribbon in spacetime, a closed string as a tube. Our job, then, is to compute amplitudes over time. What is the amplitude associated with the string having a configuration at one time and then some other confiiguration at a later time?

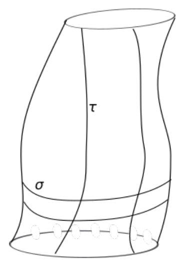

Figure 4. A closed string modeled as a tube, with parameters σ, τ.

Now, we construct an action:

$$s = \int d\tau\, d\sigma \left[\left(\frac{\partial x^\mu}{\partial \tau} \right)^2 - \left(\frac{\partial x^\mu}{\partial \sigma} \right)^2 \right], \tag{10}$$

where $\left(\frac{\partial x^\mu}{\partial \tau} \right)^2$ represents the kinetic energy and $\left(\frac{\partial x^\mu}{\partial \sigma} \right)^2$ represents the potential energy. Thus,

$$\text{Amplitude} = \int_{\text{all poss.surfaces}} e^{\, i \int d\tau\, d\sigma \left[\left(\frac{\partial x^\mu}{\partial \tau} \right)^2 - \left(\frac{\partial x^\mu}{\partial \sigma} \right)^2 \right]}. \tag{11}$$

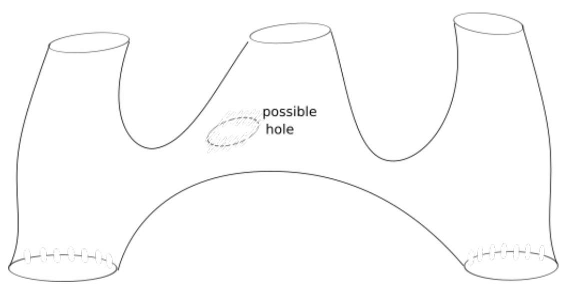

Figure 5. Two strings to three strings.

Next, we generalize: Starting with two strings what is the amplitude of ending up with three strings? Answer: Use (11).

Analogous to the previous cases, wild paths must be small. Now, let $\tau \to -i\tau$:

$$\text{Amplitude} = \int e^{- \int d\tau\, d\sigma \left[\left(\frac{\partial x^\mu}{\partial \tau} \right)^2 + \left(\frac{\partial x^\mu}{\partial \sigma} \right)^2 \right]}. \tag{12}$$

Note: Alternative string theories use the spacetime area of the tube — the Nambu-Goto Action.

We now ask what are the rules for choosing σ and τ and what are the invariants of the Lagrangian?

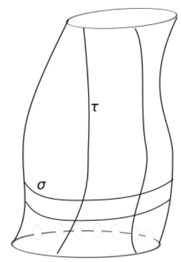

Figure 6. What are the invariants of the Lagrangian under σ and τ transformations.

The equation of motion derived by varying the action would look like a wave equation:

$$\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 + \left(\frac{\partial x^\mu}{\partial \sigma}\right)^2 = 0 \tag{13}$$

is the Laplace equation in two dimensions. The derivative goes as $x(3) - x(2)$, say for a comparison at point 2,3, or the differences. Then divide by ϵ to get the first derivative, $[\,x(3) - x(2)\,]/\epsilon$. The second derivative is like the difference of the first derivatives.

The discrete analogy.

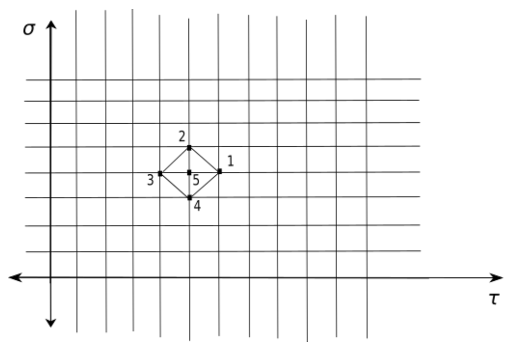

Figure 7. The discrete second derivative.

$$[\,x(3) - x(2)\,] - [\,x(2) - x(1)\,] = x(3) + x(1) - 2x(2)\,. \tag{14}$$

$$\Delta^2 x_\tau = x(1) + x(3) - 2x(5)\,. \tag{15}$$

So Eq. (13) becomes

$$x(1) + x(3) - 2x(5) + x(2) + x(4) - 2x(5) = 0\,. \tag{16}$$

Therefore,

$$x(5) = \frac{x(1) + x(2) + x(3) + x(4)}{4}\,. \tag{17}$$

However, the Laplace equation is invariant under a rotation of the σ-τ plane. So, for any infinetesimal square centered at a point, the field value at the point is the average of the field values on the corners. Hence, the invariant transformations take an infinitesimal square to some other infinitesimal square, the new coordinates being

$$\sigma' = \sigma'(\sigma,\tau)\,, \quad \tau' = \tau'(\sigma,\tau)\,. \tag{18}$$

What we have here is a conformal mapping.

Note: It's not enough to preserve only the corner angles. We have to preserve the "central" angles as well, such as formed by the diagonals of the square. The following transformation would not be conformal:

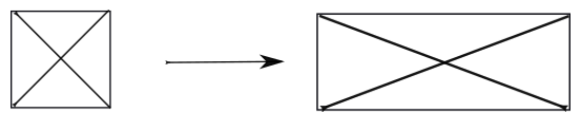

Figure 8. The central angles are not preserved. Not conformal when mapping corners to corners, but may be if mapped more cleverly.

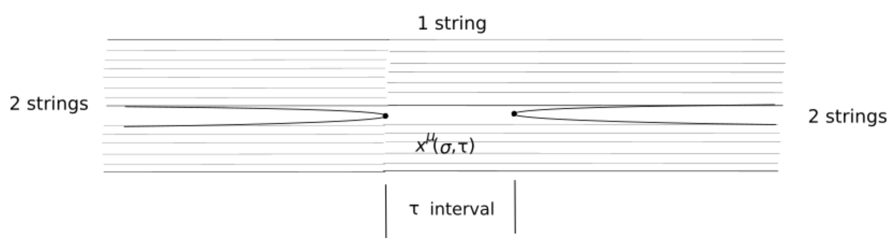

Figure 9. The x^μ satisfy the Laplace Equation.

What is the topology of the ribbon? It's a collection of concentric circles.

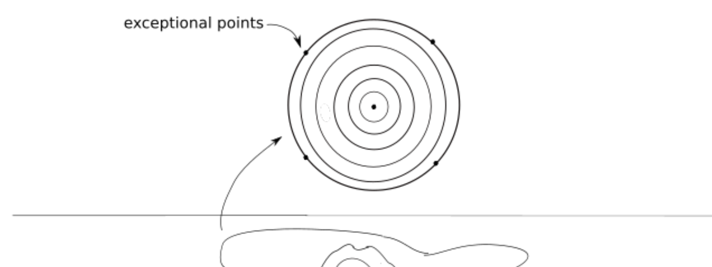

Figure 10. Conformal maps of the 2-1-2 open string to the unit disk retains the Laplace equation. The exceptional points are where 'particles' come in and go out.

Now we integrate over all the x's that live in the disk. Vertex points correspond to input-output of information. Up to a conformal transformation, the angular description of the vertices is a one-parameter mapping.

1) choose left-right symmetry,

2) choose up-down symmetry

Thus, equivalent:

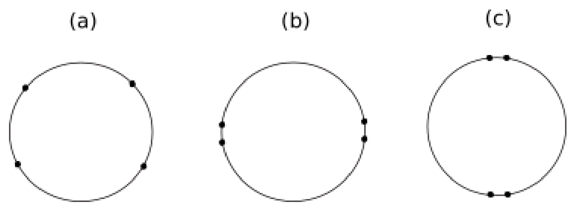

Figure 11. They are distinguished by τ intervals, the one-parameter that governs where we put the points on the rim.

It is from intergating on these injection points (vertices) that gives us the Beta function.

If the time interval is very long we get (b), that is, the intermediate state lasts a long time.

If the time interval is short, we get (c).
Now, (b) and (c) satisfy the symmetry which we have already noted as $s \leftrightarrow t$.

Figure 12. Momenta in and out.

For a review, the Mandelstam s variable has relation:

$$(k_1 + k_2)^2 = s = E_{\mathrm{CM}}^2 \,. \tag{19}$$

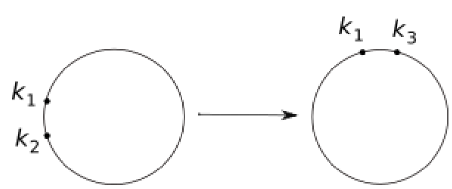

Figure 13. A Mandelstam transformation.

$$(k_1 + k_3)^2 = t = (E_{\mathrm{CM}}^2 - m^2)(1 - \cos\theta) \,, \tag{20}$$

where θ is the angle of scattering. (Remember that we set up that all the k's are incoming.)

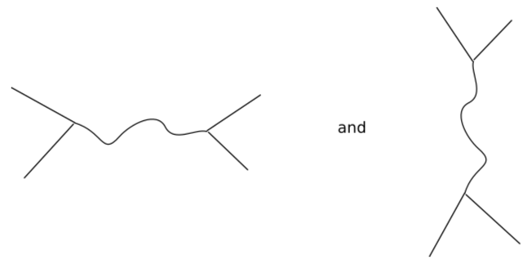

Figure 14. Left: annihilation-creation; right: emission-reabsorption.

The Feynman diagrams portrayed in Fig. 14 shows “duality” or rather “channel duality.” And we can credit this duality to conformal symmetry.