String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 8

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Abstract

This paper contains my notes on Lecture Eight of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belong solely to me.

1 Intro: Conformal Invariance

The equations of electrostatics in 2D are

$$\nabla\phi = \mathbf{E}\,, \quad \nabla\cdot\mathbf{E} = \rho \tag{1}$$

$$\nabla^2\phi = \rho \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ (where } \rho = 0) \tag{2}$$

has conformal invariance (angle preserving).

We can map region to region with solutions in the complex plane.

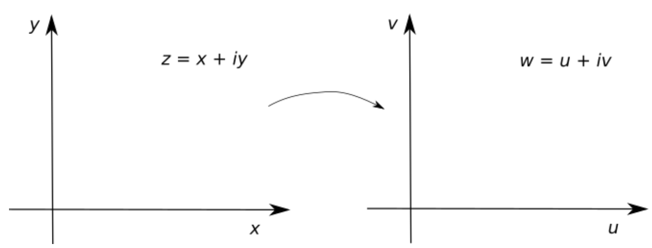

Figure 1. Complex variable mapping: $w = w(z)$.

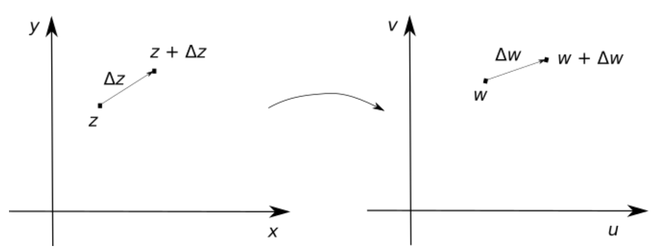

Figure 2. What can we know about $w'(z)$?

If $w'(z)$ exists, then

$$w'(z) = \lim_{\Delta z \to 0} \frac{W(z + \Delta z) - W(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z} \,. \tag{3}$$

However, for $w'(z)$ to make sense, it must have the same value regardless of which direction Δz has in the complex plane.

$$w'(z) = \frac{dW}{dz} = \frac{du + idv}{dx + idy} \,. \tag{4}$$

Since this derivative must be true for every direction in the plane, it must be true along the x axis, and also along the y axis.

$$\left.\frac{dW}{dz}\right|_{\substack{x \text{ axis} \\ dy=0}} = \frac{du + idv}{dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \,, \tag{5a}$$

$$\left.\frac{dW}{dz}\right|_{\substack{y \text{ axis} \\ dx=0}} = \frac{du + idv}{idy} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} \,. \tag{5b}$$

This brings us to the Cauchy-Riemann Equations:

$$\text{Real part} : \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,, \tag{6a}$$

$$\text{Imaginary part} : \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \,. \tag{6b}$$

Differentiating yields,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \,, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \,. \tag{7}$$

From this we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \,, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \,. \tag{8}$$

2 Polar Coordinates

Let us represent a second variation of z, different than δz, by Δz. And

$$\delta z = \rho e^{i\theta} \,, \quad \Delta z = \rho' e^{i\theta'} \,. \tag{9}$$

So,

$$\frac{\delta z}{\Delta z} = \frac{\rho}{\rho'} e^{i(\theta - \theta')} \,. \tag{10}$$

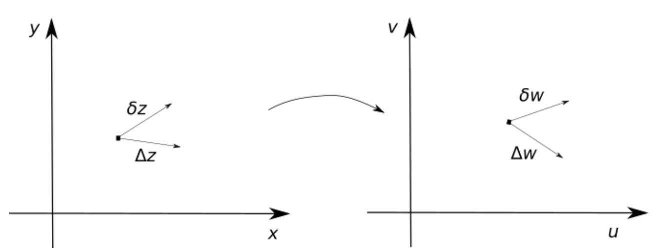

Figure 3. Two independent directions.

Now,

$$\frac{\delta w}{\Delta w} = \frac{w'\delta z}{w'\Delta z} = \frac{\delta z}{\Delta z} = \frac{\rho}{\rho'} e^{i(\theta-\theta')}\,, \tag{11}$$

giving the same difference of angle.

3 Examples:

Is $w = z^2$ analytic?

$$x^2 - y^2 + 2xyi = u + iv\,. \tag{12}$$

Then

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \tag{13a}$$

$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}\,, \tag{13b}$$

which are the Cauchy-Riemann equations.

However, if we try $w = z^* = x - iy$, we will find that this does not work.

What about $w = e^z$?

$$w = e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y\,. \tag{14}$$

Then,

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \checkmark \tag{15a}$$

$$\frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y} \quad \checkmark\,. \tag{15b}$$

Now we try $w = \log z$. Let's begin by looking at z in polar coordinates. From

$$z = r\,e^{i\theta}\,, \tag{16}$$

we get

$$\log re^{i\theta} = \log r + \log e^{i\theta} = \log r + i\theta\,, \tag{17}$$

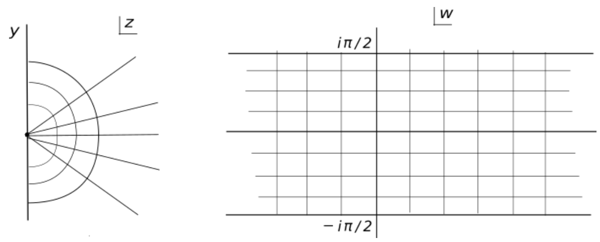

Figure 4. The $\pm y$ axis corresponds to $\theta = \pm\pi/2$. The w space is like the world sheet of a string in τ, σ coordinates. The incoming string injects at the origin of the z-plane.

Regarding the above figure, if we shift the w-plane to the left then the figure in the z-plane shrinks by a uniform factor. Push it to the right, it expands or dilates.

Now if we map from $-\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, that will include the left-half plane and will map the upper line to the lower line.

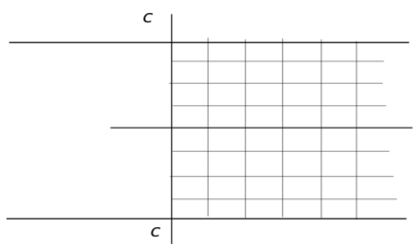

Figure 5. The resulting map is a cylinder in the w-plane.

But, since $-\pi/2 = 3\pi/2$, the points c, c' are identified with each other. Thus, we get a cylinder in w-space, which represents a closed string.

Next, a linear fractional transformation,

$$w = \frac{z+1}{z-1}\,, \tag{18}$$

which maps circles to circles and lines to lines.

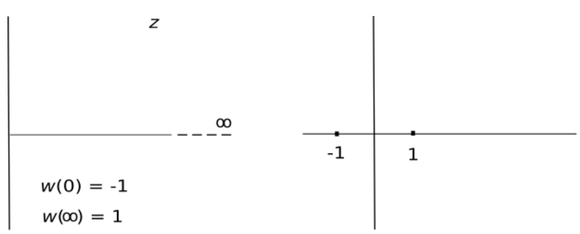

Figure 6. Special mappings.

With $z = iy$,

$$w = \frac{1+iy}{-1+iy} = -\frac{1+iy}{1-iy} = -\frac{re^{i\theta}}{re^{-i\theta}} = (-1)e^{2i\theta} = e^{2i\theta - i\pi}\,. \tag{19}$$

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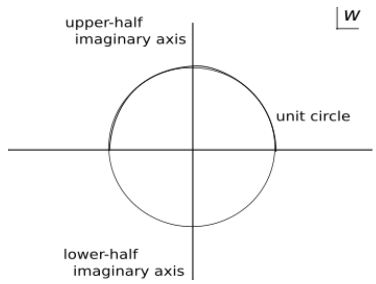

Figure 7. Thus w maps the half-plane $x \geq 0$ to the interior of the unit disk.

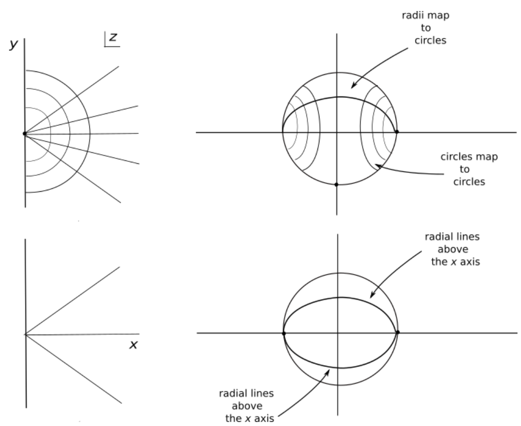

Figure 8. Thus w maps the half-plane $x \geq 0$ to the interior of the unit disk.

4 String Theory Scattering Amplitudes

Review:

Degrees of freedom of a spacetime point of a world sheet.

Calculate a path integral.

Perform the transformation (a Wick rotation ?): $\tau \to i\tau$

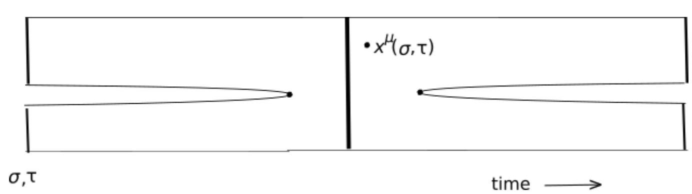

Figure 9. Two strings in, two strings out.

Next, we perform the integral:

$$\text{Amplitude} = \int_{\substack{\text{all poss. ways of filling} \\ \text{up the surface}}} e^{-i\int d\tau\, d\sigma \left[\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 + \left(\frac{\partial x^\mu}{\partial \sigma}\right)^2\right]} . \tag{20}$$

We account for the incoming/outgoing momenta "by hand." Also, we need one more integral to account for the historiesl of the particles:

$$\int dz \int e^{-i\int d\tau\, d\sigma \left[\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 + \left(\frac{\partial x^\mu}{\partial \sigma}\right)^2\right]} . \tag{21}$$

Now, we perform a conformal transformation: Map the strip to a disk.

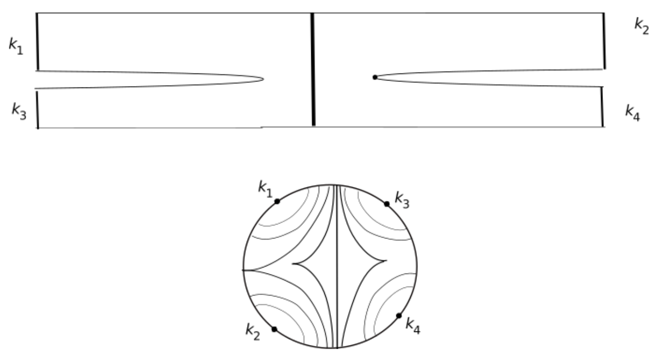

Figure 10. Conformal transformation: Map the strip to a disk. The incoming and outgoing particles are represented by points on the boundary.

For each momenta, we put into the integrand a factor of

$$\prod_{\substack{\text{particle} \\ \text{on boundary}}} e^{ikx(z)} . \tag{22}$$

Note: This is open string theory:

$$\int dz \int e^{-i\int \left[\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 + \left(\frac{\partial x^\mu}{\partial \sigma}\right)^2\right] d\tau\, d\sigma} \prod_{\substack{\text{momenta} \\ \text{on boundary}}} e^{ikx(z)} . \tag{23}$$

The z's are the positions on the disk where the momenta are 'injected'.

Then we have

$$\int e^{-\text{Action}} \prod_i e^{ik_\mu x^\mu(z_i)} . \tag{24}$$

for a total of 26 kinds of charge.

Now, think of the disk as a world in 2-D.