String Theory and M-Theory Notes for L. Susskind's Lecture Series (2011), Lecture 9

P. Reany

November 29, 2023

Abstract

This paper contains my notes on Lecture Nine of Leonard Susskind's 2011 presentation on String Theory and M-Theory for his Stanford Lecture Series. These read-along notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for all errors in these notes belongs solely to me.

1 Introduction

String theory is more constrained than the theory of point particles. Consider a particle constrained to move on a surface, as in the figure below. With only gravity and a normal force acting on the particle, the motion will be geodesic. Additional forces will complicate the motion.

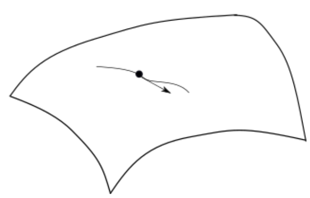

Figure 1. A particle constrained to move on a 2d surface on a trajectory.

We think of particle motion as satisfying differential equations of motion as segments on the path of motion. Thought this way, derivatives are differences added together. Then we take the limit as $\Delta t \to 0$. But how do we know that this limit exists? Classically, we know that they do, but what about in quantum mechanics?

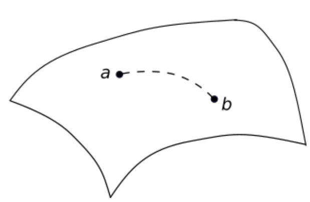

Figure 2. In quantum mechanics we are interested in the probability that the particle will arrive at b if it starts at a. For this we need a path integral.

In quantum mechanics, we ask the question: If the particle starts at a, what's the probability it will arrive at b?

There is an action principle that sums over all paths. Question: How do we know that the limit of the segment size goes to zero exists? The answer is more complicated than the previous case? The solution is equivalent to solving the Schrodinger equation. Buit even in this case, there limit exists.

Consider a sphere of radius R with a point particle constrained to move on it. Its kinetic energy is given as

$$\tfrac{1}{2}mv^2 = \frac{p^2}{2m}\,, \tag{1}$$

with the constraint that the particle must remain on the sphere.

In quantum mechanics, the momentum is quantized — linear and angular.

$$L = pR\,. \tag{2}$$

Let's rewrite the kinetic energy in terms of the angular momentum

$$\tfrac{1}{2}mv^2 = \frac{L^2}{2mR^2} = \frac{L^2}{2I}\,, \tag{3}$$

where I is the moment of inertia, which shows its importance in the particle energy levels. The amount of inertia is important for the particle energy levels.

What about the case of a string moving on a sphere? When we treat the string as a collection of individual particles, how then do we know if the limit of the string properties exists as desired? In fact, these limits do not exist.

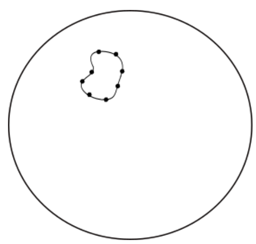

Figure 3. The case of a string constrained to move on a sphere.

A string in its ground state must have some size to accomodate its ground-state vibrations.

2 Raising and lowering to get the mean size of the ground state

I) Open strings, sizes: $X(\sigma)$

$$X(\sigma) = \sum_{\text{oscillations}} \frac{a_n^+ + a_n^-}{\sqrt{n}} \cos n\sigma\,. \tag{4}$$

In the ground state, calculate the mean size of the string. Pick an arbitrary σ. Square X.

$$X^2 = \sum_{n,m} \frac{(a_n^+ + a_n^-)(a_m^+ + a_m^-)}{\sqrt{n}\sqrt{m}} \cos n\sigma \cos m\sigma\,. \tag{5}$$

Now for the simultaeous ground states of all the oscillations (expect some zero-point energies):

$$\langle\, 0|X^2 \mid 0\,\rangle = \langle\, 0\,|\sum_{n,m}\frac{(a_n^+ + a_n^-)(a_m^+ + a_m^-)}{\sqrt{n}\sqrt{m}}\cos n\sigma\cos m\sigma\,|\,0\,\rangle\ . \tag{6}$$

Consider $a_n^+ a_m^+\,|\,0\,\rangle$, which has two units of energy and no overlap with $|\,0\,\rangle$. Now, the terms with a^- acting to the right kill $|\,0\,\rangle$. This leaves us with

$$\langle\, 0|X^2 \mid 0\,\rangle = \langle\, 0\,|\sum_{n,m}\frac{a_n^- a_m^+}{\sqrt{n}\sqrt{m}}\cos n\sigma\cos m\sigma\,|\,0\,\rangle\ . \tag{7}$$

Of course, a_m^+ will create one unit of excitment for each oscillation for the nth ocsillation. When we combine $a_n^- a_m^+$, the only nonzero terms arise when the indices are equal. Hence, we get that

$$\langle\, 0|X^2 \mid 0\,\rangle = \langle\, 0\,|\sum_{n}\frac{a_n^- a_n^+}{n}\cos^2 n\sigma\,|\,0\,\rangle\ . \tag{8}$$

But $\cos^2 n\sigma$ averages to about 1/2. Therefore,

$$\tfrac{1}{2}\,\langle\, 0\,|\sum_{n}\frac{a_n^- a_n^+}{n}\,|\,0\,\rangle = \tfrac{1}{2}\,\langle\, 0\,|\sum_{n}\frac{1}{n}\,|\,0\,\rangle\ , \tag{9}$$

where $a_n^- a_n^+\,|\,0\,\rangle = |\,0\,\rangle$. Hence

$$\langle\, X^2\,\rangle = \tfrac{1}{2}\sum_{n}\frac{1}{n} \to \infty\,. \tag{10}$$

Now, if we account for the center of mass, then

$$X(\sigma) \to X(\sigma) - X_{\rm CM}\,, \tag{11}$$

whose measure is the size of the open string. The divergence of $\langle\, X^2\,\rangle$ does not cause a real problem in flat space because of cancellation. But the cancellation do not occur on the sphere. For the time being, we will declare a maximum n, $n_{\rm max}$, and later on take the summation to infinity.

In flat space the finite open string exists in a circle of radius f and then

$$r^2 \sim \log n_{\rm max}\,. \tag{12}$$

So, r is a slowly increasing function of $n_{\rm max}$. The energy of this string is $\frac{1}{2}p^2/m$.

On to the sphere:

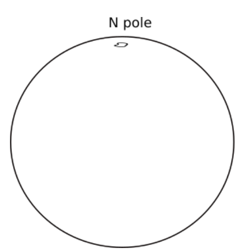

Figure 4. The string is at the North Pole. Oscillating modes cause the string to spread out. The center of mass will move along a great circle.

• If there are no oscillatory modes, the string acts like a point particle with energy $L^2/2I$, moving on a great ciurcle of a sphere. Now, if we add some modes of oscillation to the string, it takes up more space on teh surface. The more modes of oscillation, the more wiggly it becomes, though the CM of this mass will still move on the great circle.

So, if we place a rotation axis throught this sphere containing its center, so that this circle is perpemndicular to the plane of the circle the CM traverses, the different parts of this extended string are at differnet distances from this axis, affecting the overall moment of inertia of the string. As a result, quantum mechanics can treat the motion as if it were traveling along a small sphere. So, as we add more oscillating modes, the radius of this alternative sphere gets smaller, eventually down tio zero. (Renormalization group.) The moral is that string on a sphere are not good.

Note: In general, a surface is described by a Riemannian metric.

3 We now generalize the 'surface' of interest

Now, as $n_{\max}$ grows, we want the radius to eventually find a fixed point. Thus, we need to define an effective geometry for a string. At some N, the effective geometry has to stop changing.

• We need to define a Riemannian geometry for the string. For this, we need a metric tensor $g_{\mu\nu}(x)$.

• So, how does $g_{\mu\nu}(x)$ change as we add one more oscillating mode?

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \,. \tag{13}$$

This is not an issue for a flat surface, but it is for a curved surface. In a curved surface, we introduce the relation

$$\delta g_{\mu\nu}(x) = -R_{\mu\nu}(x) \,, \tag{14}$$

where $R_{\mu\nu}$ is the Ricci tensor. This has the geometry change as we add more and more fluctuations. This equation is referred to as 'Ricci Flow', which reveals the change in geometry. But we would rather fix our geometry so that it doesn't change, thus setting

$$R_{\mu\nu}(x) = 0 \,. \tag{15}$$

Besides flat spaces, which other spaces are Ricci flat ($R_{\mu\nu}(x) = 0$)? For one, $R_{\mu\nu}(x) = 0$ in GR is Einstein's vacuum field equation.

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \,, \tag{16}$$

the variable $R = R^{\alpha}{}_{\alpha}$ is called the Ricci scalar. The term on the RHS containing all the energy-momentum contributions, except from gravity. In $3+1$ spacetime, $G_{\mu\nu} = 0$ has EM waves and gravity waves for solutions. We have a set of equations to consider:

$$\begin{aligned} R_{\mu}{}^{\nu} - \tfrac{1}{2} g_{\mu}{}^{\nu} R^{\alpha}{}_{\alpha} &= 0 \,, \\ R_{\mu}{}^{\nu} - \tfrac{1}{2} \delta_{\mu}{}^{\nu} R^{\alpha}{}_{\alpha} &= 0 \,, \\ R_{\beta}{}^{\beta} - \tfrac{1}{2} \delta_{\beta}{}^{\beta} R^{\alpha}{}_{\alpha} &= 0 \,, \\ R - 2R &= 0 \,, \end{aligned} \tag{17}$$

where we used that $\delta_{\beta}{}^{\beta} = 4$ and we conclude that $R = 0$.

Therefore, the only acceptible geometry in which string theory makes sense are the solution to Einstein's solution of the field equations. However, this approach demands thye 'right' number of dimensions: Ten for the simple theory; 26 for the rival theory. In any case, the result implies gravitational waves. Thus, we require the Einstein field equations for the consistency of our version of string theory.

4 Higher dimensions

Now, how do we make sense out of the high number of dimensions? We do so through a process of compactification.

I) Superstring Theory with 10 dimensions of spacetime. We imagine a line at the macroscopic dimensions to be a thin tube in reality Along the tube, we can imagine things that can move either along the tube or around it.

II) Higher dimensions

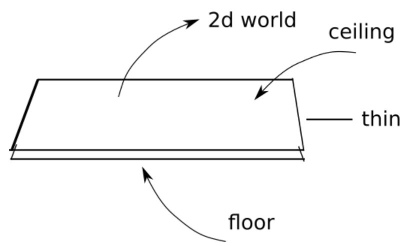

Figure 5. The space between the sheets is a hidden dimension.

We identify the ceiling and floor, compactifying 1 out of three.

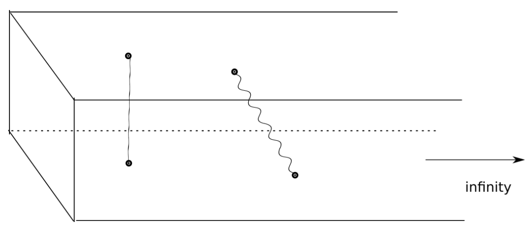

Figure 6. An infinite rectangular parallelepiped: Opposite points on floor and ceiling are identifed, and so are opposite points on the sides.

There's no edge by identifying opposite points. This process is called toroidal compactification, and it works for higher dimensions as well.

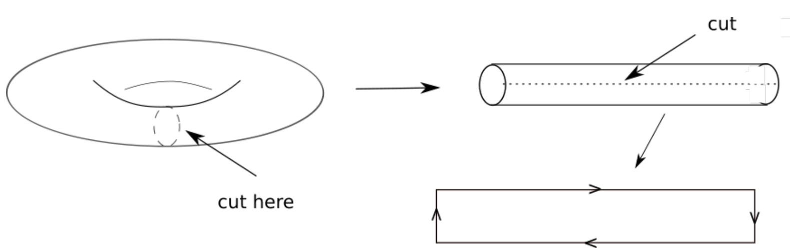

Figure 7. The torus cuts and the edge identifications.

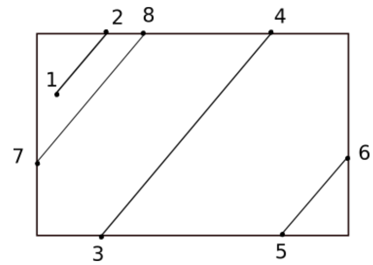

Figure 8. Motion of a particle on the 'torus'.

If the ratio of the sides are comensarate, the path will eventually overlap itself. Otherwise, it won't.

To get rid of 3 dimensions, one needs a 3-torus.

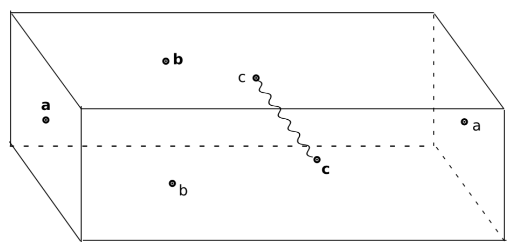

Figure 9. This will compactify 3 dimensions.

But we need to compactify 6 dimensions, so we constrain a 6-torus.

3 moduli of the torus:

1) overall size,

2) ratio of height to width,

3) angle of leaning.

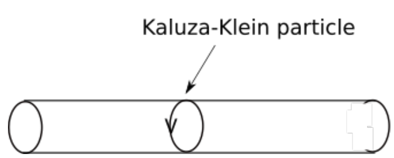

Figure 10. Kaluza-Klein particle. Circumference $2\pi r$. Think of the particle going around the tube as a small string.

$$P_c r \sim \text{angular momentum} > n\hbar.$$

$$P_c r \sim \frac{n\hbar}{r} \text{ if } \hbar \equiv 1 \text{ the } P_c \sim \frac{n}{r}.$$

If the particle is massless, $E = c|P|$. Hence, there is a quantized amount of momenta in the circumference $= m \sim 1/r$. This is where the masses of the different particles come from. The spectrum of masses is proprtional to $1/r$.

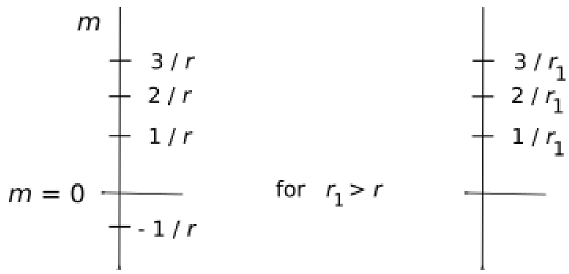

Figure 11. The bigger the circles, the closer the energy levels.

Alternatively, if the string is wrapped around the outside of the tube, it can move along the length of the tube. The mass of the particle is due to the potential energy of streching trhe string around the circumdference $\approx$ length of the string with string tension as unity (for the time being). The mass approx r / wrap. mass $\sim nr$, $n =$ number of wrap around: n is the "winding number", which can be either positive or negative. Replace $r \to 1/r$ and replace momentum by winding number. This is a symmetry of closed strings.