Statistical Mechanics Notes for L. Susskind's Lecture Series, Part 3

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Abstract

This paper contains my notes on Lecture Three of Leonard Susskind 2013 presentation on Statistical Mechanics for his Stanford Lecture Series. (He also made a 2009 video series on the same topic.) They can be found on YouTube. This time we deal with the law of entropy.

1 Getting Started

The main goal of statistical mechanics is to reproduce the equations of thermodynamics. Thermodynamics considers systems in equilibrium, which we can characterize by a few parameters, such as volume, temperature, mass, pressure, energy, etc. Now, suppose we have a given discrete system that holds its energy in discrete levels, E_i being the energy in the ith level. If there are n_i particles in the ith level, then the energy in that level is n_iE_i, and the total energy of the system is

$$\sum_i n_i E_i = E\,, \tag{1}$$

where

$$\sum_i n_i = N\,, \tag{2}$$

where N is the total number of particles of the system.

However, we can imagine our system to vary continually in time by redistributing the particles among the different energy levels, constrained by the total energy being fixed. Surely, our macroscopic thermodynamic variables are not sensitive to these random variations, or they could have never been useful.

We have two principle that will come to our rescue here. The first is the use of the statistical device of the average. Perhaps our intuition should lead us to consider the average energy of the system as more useful than its total energy, for the average energy ought to be more representative of what the majority of particles are experiencing. Thus, on dividing (1) through by N we get

$$\sum_i \frac{n_i}{N} E_i = \frac{E}{N} \equiv \bar{E}\,. \tag{3}$$

And, since $\frac{n_i}{N}$ approaches p_i for very large N, then

$$\sum_i p_i E_i = \frac{E}{N} \equiv \bar{E}\,. \tag{4}$$

We haven't yet addressed the existence of stable equilibrium of our system. To accomplish this, we'll need to concept of entropy, so let's get to that now.

2 Combinatorics, Stirling's Formula, and Entropy

We've imagined our N particles partitioned into a finite number of energy levels, there being n_i in the ith energy level. Ignoring all other constraints, in how many ways can we take those N particles and distribute them among the various energy levels? The answer comes to us from Combinatorics, as the "number of arrangements," which I'll label as Ω, but Susskind will later label as C:

$$\Omega = \frac{N!}{\Pi_i (n_i!)} \,. \tag{5}$$

At this point, we'd be quite stymied were it not for Stirling's Formula to approximate the factorials of large numbers. So, for N large,

$$N! \approx N^N e^{-N} \,. \tag{6}$$

Now, applying Stirling's Formula to (5), we get

$$\begin{aligned}
\Omega &= \frac{N^N e^{-N}}{n_1^{n_1} n_2^{n_2} \cdots} \frac{1}{e^{-n_2} e^{-n_2} \cdots} \\
&= \frac{N^N}{n_1^{n_1} n_2^{n_2} \cdots} \\
&= \frac{N^{n_1 + n_2 + \cdots}}{n_1^{n_1} n_2^{n_2} \cdots} \\
&= \frac{1}{\left(\frac{n_1}{N}\right)^{n_1} \left(\frac{n_2}{N}\right)^{n_2} \cdots} \\
&= \frac{1}{p_1^{n_1} p_2^{n_2} \cdots} \\
&= \left[p_1^{n_1} p_2^{n_2} \cdots\right]^{-1} \,.
\end{aligned} \tag{7}$$

Now, we apply the logarithm as before, to get

$$\log \Omega = -\sum_i \log p_i^{n_i} = -\sum_i n_i \log p_i \,. \tag{8}$$

This is close, but we still have that n_i to deal with. Let's divide through by N.

$$\frac{1}{N} \log \Omega = -\sum_i p_i \log p_i \,. \tag{9}$$

This then becomes our new definition of entropy S:

$$S(p_i) = -\sum_i p_i \log p_i \,. \tag{10}$$

Next time we'll introduce Lagrange multipliers and find the so-called Partition Function.