# Statistical Mechanics Notes for L. Susskind's Lecture Series, Part 4

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September 3, 2021

**Abstract**

This paper contains my notes on Lecture Four of Leonard Susskind 2013 presentation on Statistical Mechanics for his Stanford Lecture Series. (He also made a 2009 video series on the same topic.) They can be found on YouTube. This time we define the Partition Function and calculate the average energy of an ideal gas.

## 1 Getting Started

Rather than maximizing the entropy, Susskind prefers to minimize the negative entropy:

$$-S(p_i) = \sum_i p_i \log p_i \,. \tag{1}$$

Now we proceed to minimizing the negative entropy with constraints, using a Lagrange multiplier for each of the two constraints.

$$F'(p_i) = \sum_i p_i \log p_i + \alpha\big[\sum_i p_i - 1\big] + \beta\big[\sum_i E_i p_i - \bar{E}\big] \,, \tag{2}$$

which constrains $F'$ by the conservation of probability and energy. Now,

$$\frac{\partial F'(p_i)}{\partial p_i} = (\log p_i + 1) + \alpha + \beta E_i = 0 \,. \tag{3}$$

This last equation can be solved for $p_i$:

$$p_i = e^{-(1+\alpha)} e^{-\beta E_i} \,. \tag{4}$$

At this point it is customary to introduce the variable $Z$, called the *Partition Function*, by

$$Z = e^{(1+\alpha)} \,. \tag{5}$$

Thus (4) becomes

$$p_i = \frac{1}{Z} e^{-\beta E_i} \,. \tag{6}$$

Now, we haven't fully expended the usefulness of our two constraints, which is fortunate because we need them to determine the Lagrange multipliers $\alpha$ and $\beta$. For starters, $\sum_i p_i = 1$ implies that

$$\frac{1}{Z} \sum_i e^{-\beta E_i} = 1 \,. \tag{7}$$

Therefore,

$$Z = \sum_i e^{-\beta E_i}\,. \tag{8}$$

Furthermore, since $\sum_i p_i E_i = \bar{E}$ then

$$\frac{1}{Z}\sum_i e^{-\beta E_i} E_i = \bar{E}\,. \tag{9}$$

On differentiating (8) by $\beta$, we get

$$\frac{\partial Z}{\partial \beta} = \sum_i (-E_i) e^{-\beta E_i}\,. \tag{10}$$

Combining this last equation with (9) yields

$$\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \log Z}{\partial \beta}\,. \tag{11}$$

We'll come back to this result.

$$\begin{aligned} S &= -\sum_i p_i \log p_i \\ &= -\sum_i \frac{1}{Z} e^{-\beta E_i}\big[-\beta E_i - \log Z\big] \\ &= \sum_i \frac{1}{Z} e^{-\beta E_i}\big[\beta E_i + \log Z\big] \\ &= \beta \sum_i p_i E_i + \frac{1}{Z}\log Z \sum_i e^{-\beta E_i}\,. \end{aligned} \tag{12}$$

From this have

$$S = \beta \bar{E} + \log Z\,. \tag{13}$$

Now, back to temperature. From an earlier lecture we determined that

$$\frac{dS}{d\bar{E}} = \frac{1}{T}\,. \tag{14}$$

On taking the differential of (13), we have that

$$dS = \beta d\bar{E} + \bar{E} d\beta + \frac{\partial \log Z}{\partial \beta}\, d\beta = \beta d\bar{E}\,, \tag{15}$$

where we used Eq. (11). On comparing this last equation with (14), we find that

$$T = \frac{1}{\beta}\,. \tag{16}$$

Therefore, our formal Lagrange multiplier $\beta$ has taken on a direct physical meaning.

## 2 Average Energy of an Ideal Gas

In an ideal gas, all energy is kinetic. We assume that this gas is contained in a box of volume $V$. We begin with the partition function $Z$.

$$Z = \int d^{3N}x\, d^{3N}p\, e^{-\beta \sum_{n=1}^{3N} p_n^2} = V^N \int d^{3N}p\, e^{-\beta \sum_{n=1}^{3N} p_n^2} . \tag{17}$$

Here, Susskind describes a controversy about whether or not one should include a factor of $1/N!$ in this integrand, but does so himself for later simplicity.[1]

$$Z = \frac{V^N}{N!} \Big[ \int dp\, e^{-\beta p_n^2} \Big]^{3N} . \tag{18}$$

The integrand $\int dp\, e^{-\beta p_n^2}$ is just a number.

Since the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}\,, \tag{19}$$

then

$$\begin{aligned} Z &= \frac{V^N}{N!}\Big[\frac{2m\pi}{\beta}\Big]^{3N/2} \\ &\approx \frac{V^N}{N^N e^{-N}}\Big[\frac{2m\pi}{\beta}\Big]^{3N/2} \\ &= \Big(\frac{e}{\rho}\Big)^N \Big[\frac{2m\pi}{\beta}\Big]^{3N/2} , \end{aligned} \tag{20}$$

where $\rho = N/V$.

On taking the logarithm of both sides, we get

$$\begin{aligned} \log Z &= N\Big(\tfrac{3}{2}\log\frac{2m\pi}{\beta} - \log\frac{\rho}{e}\Big) \\ &= N\Big(\tfrac{3}{2}\log\frac{2m\pi}{\beta} - \log\rho + 1\Big) \\ &= -\frac{3N}{2}\log\beta + \text{const}\,. \end{aligned} \tag{21}$$

Hence,

$$\bar{E} = -\frac{\partial \log Z}{\partial \beta} = \frac{3NT}{2} . \tag{22}$$

Hence, the average energy per particle is $\dfrac{3T}{2}$.

[1] In the process of taking the logarithmic derivative of $Z$, all of its constant factors go to zero, so we can add in constant factors for free.