Statistical Mechanics Notes for L. Susskind's Lecture Series, Part 7

P. Reany

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Abstract

This paper contains my notes on the seventh lecture of Leonard Susskind's 2013 Statistical Mechanics Stanford Lecture Series. (He also made a 2009 video series on the same topic.) They can be found on YouTube. Our goal in this paper is to investigate the speed of sound in a gas.

1 Getting Started

We start off interested in calculating the speed of sound in a gas. Well, what's the average energy of a gas molecule, each having mass m?

$$\bar{E} = \tfrac{3}{2}k_B T = \tfrac{1}{2}m\overline{v}^2 \,. \tag{1}$$

Solving for $\overline{v}^2$ we get

$$\overline{v}^2 = k_B T/m \,. \tag{2}$$

This $\overline{v}$ is the approximate speed of sound in the gas

A better approximation for the speed of sound in a gas is given by

$$\frac{\partial P}{m\partial\rho} = c^2 \,, \tag{3}$$

where ρ is the number of particles in volume V. In the case of an ideal gas,

$$PV = Nk_B\mathcal{T} \,, \tag{4}$$

or

$$P = \rho k_B\mathcal{T} \,, \tag{5}$$

then

$$\frac{\partial P}{m\partial\rho} = k_B\mathcal{T} \,, \tag{6}$$

Therefore,

$$c^2 = \frac{k_B\mathcal{T}}{m} \,. \tag{7}$$

2 Harmonic Oscillators

What is the average energy of a harmonic oscillator?

$$H = \frac{P^2}{2m} + \frac{kx^2}{2}\,. \tag{8}$$

Now for the Boltzmann distribution $e^{-\beta p^2/2m} e^{-\beta k x^2/2}$:

$$Z = \int_{p=-\infty}^{\infty} e^{-\beta p^2/2mM}\,dp \int_x e^{-\beta k x^2/2)} dx\,, \tag{9}$$

where the first integral on the RHS yields

$$\sqrt{\frac{2m\pi}{\beta}}\,, \tag{10}$$

and the second yields

$$\sqrt{\frac{2\pi}{\beta k}}\,. \tag{11}$$

Therefore,

$$Z = \frac{2\pi}{\omega}\frac{1}{\beta}\,, \tag{12}$$

where we have used that $\omega = (k/m)^{1/2}$.

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On our way to calculating the average energy, we calculate $\log Z$:

$$\log Z = \log\frac{2\pi}{\omega} - \log\beta\,. \tag{13}$$

Hence,

$$\bar{E} = -\frac{\partial \log Z}{\partial\beta} = \frac{1}{\beta} = T\,. \tag{14}$$

3 Quantum Mechanical Harmonic Oscillators

Quantum Mechanical Harmonic Oscillator energy levels go as

$$E_n = n\hbar\omega\,. \tag{15}$$

So, the partition function goes as:

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega} = \sum_{n=0}^{\infty} [e^{-\beta\hbar\omega}]^n \\ &= \frac{1}{1 - e^{-\beta\hbar\omega}}\,, \end{aligned} \tag{16}$$

Now, on to the energy:

$$\begin{aligned} \bar{E} &= -\frac{\partial \log Z}{\partial\beta} \\ &= \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \\ &= \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\,. \end{aligned} \tag{17}$$

Let's now examine the classical limit $T \to \infty$:

$$\underset{T\to\infty}{\bar{E}} \overset{\beta\to 0}{=} \frac{\hbar\omega}{\beta\hbar\omega} = \frac{1}{\beta} = T\,. \tag{18}$$

Let's now examine the low-temperature limit $T \to 0$:

$$\underset{T\to 0}{\bar{E}} \overset{\beta\to\infty}{=} \hbar\omega e^{-\beta\hbar\omega} \to 0\,. \tag{19}$$

Hence, very low temperatures tend to suppress any oscillation modes to excite.

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What's the boundary value separating the classical from quantum behavior of this sysem?

$$\beta\hbar\omega \approx 1 \qquad \text{or} \quad \hbar\omega = T\,. \tag{20}$$

Thus,

$$\beta\hbar\omega > 1 \qquad\qquad \text{quantum region}\,, \tag{21a}$$

$$\beta\hbar\omega < 1 \qquad\qquad \text{classical region}\,. \tag{21b}$$

Again, if $T < \hbar\omega$, that's less than one quantum of energy and therefore an oscillation will not be excited. Thus, a diatomic molecule at this temperature will act like a monotonic molecule.

4 Entropy and Liouville's Theorem

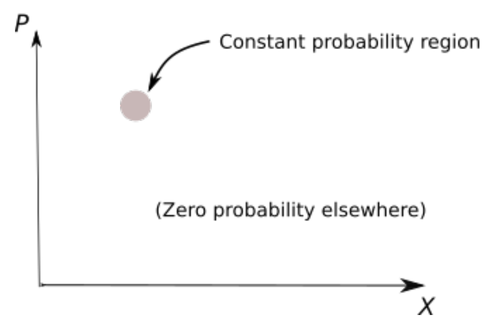

Figure 1. Phase space diagram. The circle (the 'bold') is a region of constant probability. The blob will deform in time so as to keep its volume fixed.

By Liouville's Theorem, the blob will deform (evolve) in time so as to keep its volume fixed. Course graining attempts to chop up phase space to produce a finite number of volume elements.