Statistical Mechanics Notes for L. Susskind's Lecture Series, Part 8

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September 3, 2021

Abstract

This paper contains my notes on the eight lecture of Leonard Susskind's 2013 Statistical Mechanics Stanford Lecture Series. (He also made a 2009 video series on the same topic.) They can be found on YouTube. Our goal in this paper is to introduce the **Ising Model**.

1 Getting Started

I'm starting my notes at about 43:00 time stamp.

Boltzmann: 1844 – 1906
Poincare: 1854 – 1912

Assume we have N atoms in a row, all aligned magnetically, either up or down. Furthermore, although there is an external magnetic field μ affecting their alignments, they do not affect each other. Each atom has a magnetic moment μH.

$$\begin{cases} \text{every up gives} & +\mu H \\ \text{every down gives} & -\mu H \end{cases} \tag{1}$$

Let $\sigma(1)$, $\sigma(2)$, ..., represent the orientations of the atoms. $\sigma(i) = \pm 1$. Let n be the number of ups and m be the number of downs. Then

$$n + m = N\,. \tag{2}$$

The energy of the system is then

$$E = (n - m)\mu H\,. \tag{3}$$

Given the pair (n, m), what are the number of states?

$$\text{No. of states} = \binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{N!}{n!m!}\,. \tag{4}$$

Then

$$Z = \int_{n,m} e^{-\beta\mu H(n-m)}\, dp\,. \tag{5}$$

For convenience, let

$$X = e^{-\beta\mu H}\,, \tag{6}$$

$$Y = e^{\beta\mu H}\,, \tag{7}$$

$$\tag{8}$$

then

$$Z = \sum_n \frac{N!}{n!(N-n)!} X^n Y^m = (X+Y)^N \,, \tag{9}$$

by use of the Binomial expansion.

Hence,

$$\begin{aligned} Z &= (e^{-\beta\mu H} + e^{\beta\mu H})^N && (10)\\ &= 2^N (\cosh\beta\,\mu H)^N \,. && (11) \end{aligned}$$

Of interest to us is the magnetization M of the system, which is proportional to the difference $n-m$.

$$M \equiv \frac{n-m}{N} \,. \tag{12}$$

Next,

$$E = NM\mu H \,, \tag{13}$$

Finally,

$$\bar{E} = N\langle\, M\,\rangle \mu H = N\mu H\overline{M} \,, \tag{14}$$

Now,

$$\log Z = N\log(\cosh\beta\,\mu H) + \text{const} \,. \tag{15}$$

Therefore,

$$\bar{E} = -\frac{\partial \log Z}{\partial\beta} = -N\mu H(\tanh\beta\,\mu H) \,. \tag{16}$$

On eliminating $\bar{E}$ between (14) and (16), we get

$$\overline{M} = -\tanh\beta\,\mu H \,. \tag{17}$$

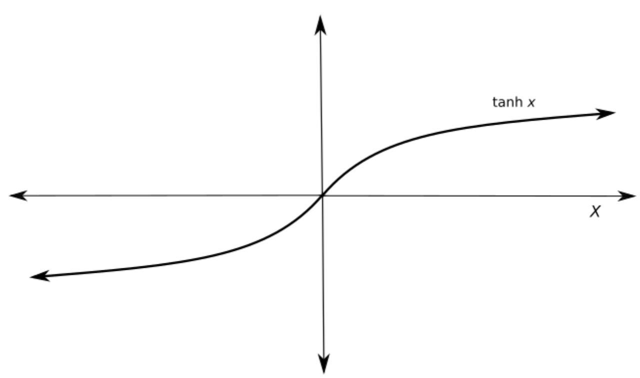

Figure 1. Graph of $\tanh x$.

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For low temperatures, β is large, $\overline{M} = -\tanh\beta\,\mu H \to -1$ (all spins are down). For high temperatures,

$$\beta \to 0, \quad \overline{M} \to 0 \,, \tag{18}$$

because the excessive heat randomizes the spin orientations to produce and average of zero.

Time Stamp: 1:16:00

2 One-dimensional Ising Model

1) No external magnetic field.

2) Each atom will experience the magnetic fields only of its neighbors.

Neighbors are said to be "in alignment" if they are either both up or both down. Otherwise, they are not "in alignment." Here's the corresponding energy rule: If the neighbors are aligned, the energy is lower; if they are unaligned, the energy is higher.

We imaging our magnetic atom in a 1-d lattice.

$$E_{12} = -j\sigma(1)\sigma(1) \tag{19}$$

$$E_{n,n+1} = -j\sigma(n)\sigma(n+1)\,, \tag{20}$$

where the σ's are the magnetic moments. At high energies, the spin directions are randomized and thus there is no magnetization.

Note: There is a global symmetry at play, if for all indices i

$$\sigma(i) \rightarrow -\sigma(i)\,. \tag{21}$$