Statistical Mechanics Notes for L. Susskind's Lecture Series, Part 9

P. Reany

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Abstract

This paper contains my notes on the ninth lecture of Leonard Susskind's 2013 Statistical Mechanics Stanford Lecture Series. (He also made a 2009 video series on the same topic.) They can be found on YouTube. Our goal in this paper is to further investigate the Ising model for its ability to describe phase transitions.

1 Getting Started

This time we change our perspective. We will think of our system as a 'small system' in a heat bath. So, we will pick out a single magnet and treat it as being in equlibrium with its surroundings.

$$E = -\mu B \sigma \,. \tag{1}$$

The partition function of the system is then

$$Z = \sum_{\sigma=\pm 1} e^{+jB\sigma} = e^{+jB} + e^{-jB} = 2\cosh \beta j \,. \tag{2}$$

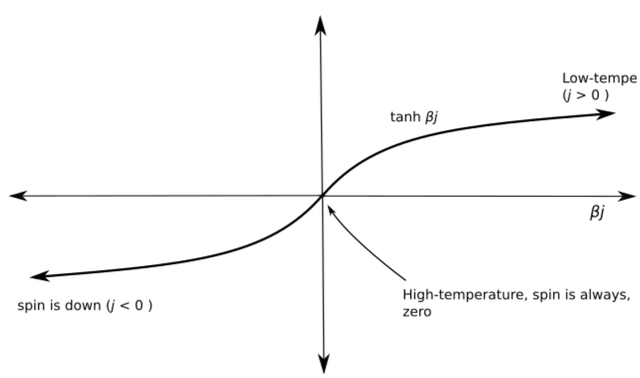

Figure 1. Graph of $\tanh \beta j$.

Since each point is independent of all others,

$$Z_{\text{total}} = e^{+jB} + e^{-jB} \,. \tag{3}$$

Then

$$\bar{E} = -\frac{1}{Z}\frac{\partial \log Z}{\partial \beta} = -\frac{2j}{Z}\sinh\beta j = -j\tanh\beta j\,. \tag{4}$$

So, the magnetization goes as

$$\langle\,\sigma\,\rangle = \tanh\beta j\,. \tag{5}$$

Now we return to the 1-d Ising model with no external magnetic fields.

$$\bar{E} = -j\sum \sigma_i\sigma_{i+1}\,, \tag{6}$$

where the minus sign means that it favors being parallel. In the ground state, this means that the spins are aligned either up or down (this is a symmetry of the system).

$$Z = \sum_i^{\infty} e^{-j\beta\sum\sigma_i\sigma_{i+1}}\,. \tag{7}$$

Now we consider a conditional correlation function:

> Given a particular magnet orientation, what is the probability that at some position of magnets down the line, it has a certain orientation?

What is the average of the product of spins at two different locations?

If the spin at location $i+n$ is affected by the spin value at location i, then $\langle\,\sigma_i\sigma_{i+1}\,\rangle$ would not be zero.

Case 1) We begin analysis of a long chain of spins with the first one up. Since we know that σ_1 is up, then the product $\sigma_1\sigma_2$ will tell us everything we want to know about σ_2.

Let's introduce a new variable μ that will represent the bonds between adjacent particles, thus

$$\mu_1 = \sigma_1\sigma_2\,, \quad \mu_2 = \sigma_2\sigma_3\,, \quad \text{and so on.} \tag{8}$$

Hence $\mu_i = \pm 1$, $+1$ for aligned neighbors, -1 for unaligned neighbors. Therefore,

$$E = -j\sum_i \mu_i\,. \tag{9}$$

Note: There are no relationships between the μ's.

$$Z = 2\sum_{\mu_i} e^{-j\sum\beta\mu_i}\,, \tag{10}$$

where the factor of 2 comes from the facts that we could have started with the first one down. So,

$$Z = (2\cosh j\beta)^{N-1}\,, \tag{11}$$

where $N-1$ is the number of bonds.

Now,

$$\langle\,\mu\,\rangle = \langle\,\sigma_i\sigma_{i+1}\,\rangle = \tanh\beta j > 0 \quad \text{because } j > 0\,. \tag{12}$$

Therefore, there is a net tendency for particles i and $i+1$ to align.

The correlation of σ_i with σ_{i+n}

$$\langle\,\sigma_i\sigma_{i+n}\,\rangle = \langle\,\sigma_i\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{i+n}\,\rangle = \langle\,\mu_1\mu_2\cdots\mu_{n-1}\,\rangle\,. \tag{13}$$

But since the μ's are independent, then

$$\langle\,\sigma_i\sigma_{i+n}\,\rangle = \left\langle\,\prod_{i=1}^{n-1}\mu_i\,\right\rangle = (\tanh\beta j)^{n-1}\,. \tag{14}$$

2 Mean Field Approximation

We move now to treat the subject in d dimensions. We pick an embedded particle in a lattice. We're interested with how it interacts with its neighbors. I will refer to it as the "central object" or CO.

$$E = -j\sigma \sum_{\substack{\text{neigh-}\\\text{bors}}} \sigma \,, \tag{15}$$

Now, assume that the average spin is $\bar{\sigma}$. Then,

$$E \approx -j(2d)\sigma\bar{\sigma} \,, \tag{16}$$

The average spin of the CO is given by $\bar{\bar{\sigma}}$. Therefore,

$$\bar{\bar{\sigma}} = \tanh 2\beta dj\bar{\sigma} \,, \tag{17}$$

where the average magnetization $\bar{\bar{\sigma}} = \overline{M}$, from Eq. (17), page 2 of Part 8. But we must insist that $\bar{\bar{\sigma}}$ is no different than its neighbors,[1] hence

$$\bar{\bar{\sigma}} = \bar{\sigma} \,, \tag{18}$$

therefore,

$$\bar{\sigma} = \tanh 2\beta dj\bar{\sigma} \,. \tag{19}$$

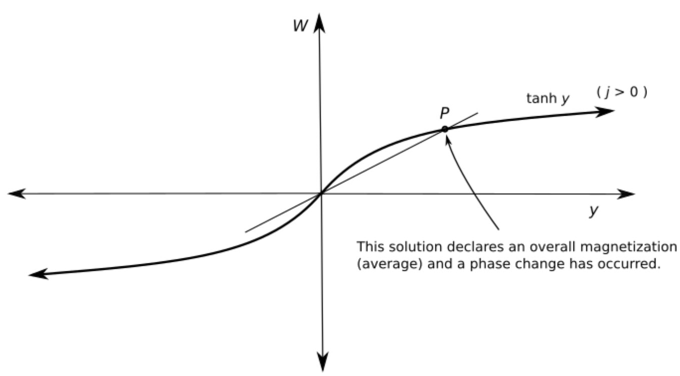

Figure 2. Graph of $\tanh y$, where $y = 2\beta dj\bar{\sigma}$. The variable W is a dummy variable to label the vertical axis.

We make the change in variables

$$y = 2\beta dj\bar{\sigma} \,. \tag{20}$$

Thus, (19) becomes

$$\frac{y}{2\beta dj} = \tanh y \,, \tag{21}$$

Now, we have the two coupled equations to solve simultaneously, either graphically or numerically:

$$W = \frac{1}{2\beta dj} y = \frac{T}{2dj} y \,, \tag{22a}$$

$$W = \tanh y \,. \tag{22b}$$

A straight line through the origin of slope 1 will only intersect the $\tanh y$ function at the origin. But as soon as the slope is greater than 1 or less than 1, the line will also intersect the $\tanh y$ function other than at the origin. The point P in Figure 2 is one such intersection point.

[1] The Mean Field Approximation is also called the Self-Consistency Field Theory.