Stat. Mech. Notes for L. Susskind's Lecture Series (2009), Lecture 7

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Abstract

This paper contains my notes on Lecture Seven of Leonard Susskind's 2009 presentation on Statistical Mechanics for his Stanford Lecture Series. The following notes are meant to supplement the more extensive notes I made for his 2013 Lecture series. This time we look at Black-Body Radiation.

1 Getting Started

The energy levels of a quantum mechanical oscillator go as

$$E_n = \hbar\omega n\,. \tag{1}$$

Imagine that we have a box at temperature T, containing a superposition of harmonic oscillators at various wavelengths. According to classical thermodynamics, each oscillator in thermal equilibrium should have an energy equal to kT. But classically there seemed to be no limit on the smallest wavelength, hence the energy in all the wavelengths should blow up.

Early radiation studies coming from a black box indicated that there existed a shortest wavelength, and it was a function of temperature.

Planck's solution attempt was to assume that the chamber of the black box was surfaced by atoms that emitted and absorbed radiation only at definite frequencies. He then proposed that when an atom oscillates, it emitted EM energy at discrete wavelengths ω, given by (1), technically,

$$E_n = nh\nu\,. \tag{2}$$

A better way to think of this is that the radiation is quantized at discrete frequencies. Later on, Einstein would interpret this radiation as photons with quantized frequencies.

For the radiation in the box to reflect, we need the E-field to be zero on the walls.

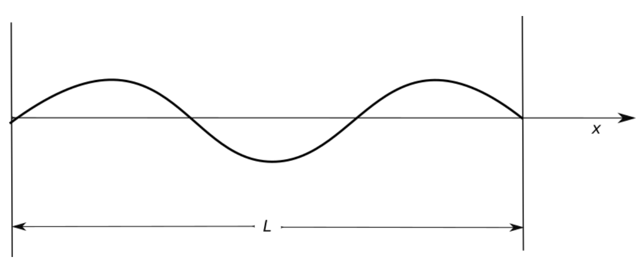

Figure 1. The acceptable EM waves have to fit into the cavity so that the E-field at the walls is zero.

$$Y = Y_m \sin \frac{m\pi}{L} x \,, \qquad m = 0, 1, 2, 3, \ldots \,. \tag{3}$$

On summing,

$$Y(x) = \sum_m Y_m \sin \frac{m\pi}{L} x \,. \tag{4}$$

We can re-express the arguments in terms of the wavenumbers

$$Y_m \sin \frac{m\pi}{L} x \to Y_m \sin kx \qquad \text{where} \qquad k \equiv \frac{m\pi}{L} \,. \tag{5}$$

If we define c as the speed of propagation of the wave, then

$$\omega = ck \,. \tag{6}$$

Now we consider the waves that can propagate inside the chamber in three-dimensions.

$$F = \sum Y_m \sin \frac{m_x\pi}{L} x \;\; \sin \frac{m_y\pi}{L} y \;\; \sin \frac{m_z\pi}{L} z \,, \tag{7}$$

where m_x, m_y, m_z are integers when the cavity is a cubic box of dimensions $L \times L \times L$. Or, we write (8) in terms of wavenumbers:

$$F = \sum Y_m(t) \sin k_x x \;\; \sin k_y y \;\; \sin k_z z \,, \tag{8}$$

and now

$$\omega = c|\,\mathbf{k}\,| = c\sqrt{k_x^2 + k_y^2 + k_z^2} \tag{9}$$

is the frequency of an oscillation.

Now we ask how much energy is stored in all the waves. The radiation field inside the box comes into thermal equilibrium with the walls of the box, which are at temperature T. Let's look at the energy stored in a wave of wavenumber k.

$$E = \frac{\omega\hbar}{e^{\beta\omega\hbar} - 1} = \frac{|\,\mathbf{k}\,|c\hbar}{e^{\beta|\,\mathbf{k}\,|c\hbar} - 1} \,. \tag{10}$$

By summing on all the wavenumbers, we get

$$E = \sum_{m_x, m_y, m_z} \frac{|\,\mathbf{k}\,|c\hbar}{e^{\beta|\,\mathbf{k}\,|c\hbar} - 1} \,. \tag{11}$$

We will approximate this sum with an integral. If L is large, the difference between neighboring values of k is small. So, for $k = m\pi/L$

$$\Delta k_x = \Delta k_y = \Delta k_z = \frac{\pi}{L} \,. \tag{12}$$

Then

$$\Delta^3 k \sum_{m_x, m_y, m_z} \to \int dk_x \, dk_y \, dk_z \,. \tag{13}$$

Hence,

$$\sum_{m_x, m_y, m_z} \to \frac{1}{\Delta^3 k} \int dk_x \, dk_y \, dk_z \,. \tag{14}$$

Therefore, our expression for the total energy is

$$E = \frac{L^3}{\pi^3} \int \frac{d^3k |\,\mathbf{k}\,| c\hbar}{e^{\beta |\,\mathbf{k}\,| c\hbar} - 1} \,. \tag{15}$$

With change of variables

$$\begin{aligned} u/\beta c\hbar &= k \,, \\ k_x c\hbar &= u_x \,, \\ k_y c\hbar &= u_y \,, \\ k_z c\hbar &= u_z \,, \end{aligned} \tag{16}$$

then,

$$\begin{aligned} E &= \frac{L^3}{\pi^3} \int \frac{d^3k |\,\mathbf{k}\,| c\hbar}{e^{\beta |\,\mathbf{k}\,| c\hbar} - 1} \\ &= \frac{L^3}{\pi^3} \int \frac{d^3u}{(\beta c\hbar)^3} \frac{|\,\mathbf{u}\,|}{\beta} \frac{1}{e^u - 1} \\ &= \frac{L^3}{\pi^3} \frac{1}{c^3\hbar^3\beta^4} \int_0^\infty d^3u \, \frac{u}{e^u - 1} \,. \end{aligned} \tag{17}$$

On converting the integral, which is in rectangular coordinates to spherical coordinates, we get

$$\int_0^\infty d^3u \, \frac{u}{e^u - 1} \to 4\pi \int_0^\infty dr \, \frac{r^3}{e^r - 1} \,. \tag{18}$$

Now we divide this integral by eight because we want to restrict the values of the wavenumbers to only nonnegative values, getting

$$\int_0^\infty d^3u \, \frac{u}{e^u - 1} \to \frac{4\pi}{8} \int_0^\infty dr \, \frac{r^3}{e^r - 1} \,. \tag{19}$$

Hence,

$$E_{\text{total}} = \frac{L^3}{\pi^3} \frac{1}{c^3\hbar^3\beta^4} \frac{4\pi}{8} \int_0^\infty dr \, \frac{r^3}{e^r - 1} \,. \tag{20}$$

But the definite integral $\displaystyle\int_0^\infty dr \, \frac{r^3}{e^r - 1}$ has the value $\pi^4/15$.

Finally, we add in a factor of 2 to account for two polarizations, and we replace β^{-1} by T, to get

$$E_{\text{total}} = \frac{\pi^2}{15\hbar^3c^3} \frac{L^3}{\beta^4} = VT^4 \frac{4}{c} \sigma \,, \tag{21}$$

where σ is the Stephan-Boltzmann constant.

Now, there is a natural cutoff frequency ω_{cutoff} that depends on the temperature at which an oscillator can no longer function, and that is when the oscillator energy goes as

$$\hbar\omega_{\text{cutoff}} = T \,, \tag{22}$$

the classical enbergy. Thus the cutoff frequency is

$$\omega_{\text{cutoff}} = \frac{T}{\hbar} \,. \tag{23}$$